Mathematics as a resource in understanding the peculiar characteristics of magnetic field

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Abstract
Despite that mathematics is the basic language used by physicists to construct formal entities, at high school level sometimes this is poorly implemented, as the case of the naïve definitions of important entities like the pseudovector (axial vector) and the limited use of the study of the symmetries for example in the analysis of electromagnetic systems. In this paper two simple experimental contexts are proposed in which students investigate the pseudo-vectorial nature of electromagnetic field vector and a simple formal explanation of its nature starting from the analysis of its behavior.

Introduction
Mathematics provides to physicists powerful ways to describe the phenomenological world through formal entities, with properties that constitute tools for the analysis and allows researchers to deduce important conclusions starting from the individuation of simple elements observable into the physical systems. In particular, in this framework, one of the most important theoretical tool is the Neother’s first theorem (Noether et al 1918) that, in its simplest formulation, relates the presence of symmetries in a physical system to conservation laws: symmetries in classical and modern physics have a pivotal role in the description of the physical systems, so the knowledge of how formal entities are transformed by symmetry operation is crucial (Foot & Volkas, 1995; Kozlov & Valerij, 1995; Mohapatra & Senjanovi´c, 1981; Redlich, 1984).

The role of symmetries in high school physics education is often underestimated and this is due to a not so strictly definition of the formal entities, that points only to the definition of the structure of the entities and not to the way in which these entities are transformed by symmetry operations. The main example is the definition of ‘vector’, without stressing the difference between polar and axial vector. This distinction becomes relevant only in the higher level courses, creating so an intellectual gap between student’s studies in undergraduate and graduate mathematics (Kolecki, 2002). In the student learning path, this gap seems to be a “no man’s land” and represents a huge difficulty for students, so a specific activity that allows students to face the difference between axial and polar vectors has to be introduced.

To analyze the symmetries of the electromagnetic system is pivotal to know the transformation properties of the electromagnetic quantities under space inversion, charge conjugation and time reversal (Rosen, 1973). In particular Pierre Curie (1894) was one of the first scientists that demonstrate that the electric and magnetic vectors are transformed in a different ways under the space inversion highlighting the different nature of the two vectors: electric field vector transforms as a “normal vector” (as position, velocity and acceleration vectors) and the magnetic field vector (Roche, 2001) transform as an axial vector -or pseudovector- (as angular momentum vector).
In this work, that is a part of a larger work of research, we highlight the role of the formalism in the description of a quantity such as the magnetic field vector $\vec{B}$, proposing two experimental context in which its pseudovectorial nature its explored.

**Experimental exploration of the pseudovectorial nature of the magnetic field vector**

We propose two contexts to introduce at high school student level the idea of the magnetic field vector as a pseudovector entity: the study of the magnetic field generated by a coil and the study of the effect of the magnetic field on a moving point charge (Lorenz force).

In the case of the magnetic field generated by a coil, experimentally, we observe experimentally that a compass set in the center of the coil indicates the presence of a magnetic field having direction coincident with the axis of the loop and the versus given by the right-hand rule.

In a reference system with the $xy$ plane parallel to the plane of the coil, we investigate the transformation properties of the formal entities describing the system.

In particular, let us consider the magnetic field vector in the center of the coils ($\vec{B}$) and a general position vector $\vec{p}$ (as shown in Figure 1a). After rotating the system around the $x$ axis, the compass shows that the magnetic field vector in the centre of the coils transforms in the same way of the position vector (Figure 1b).

![Figure 1: Rotation of a coil](image1.png)

If we consider instead a specular reflection transformation of the system respect to the $xy$ plane, the compass shows that the transformation rule of the magnetic field under this symmetry transformation differs from the transformation rule for the position vector (Figure 2b).

![Figure 2: Symmetry transformation of coil](image2.png)

The case of the Lorentz force can be experimentally explored considering a moving charge between two Helmholtz coil. If any dissipation phenomena is neglected, the motion of charge can be one of the following three types: circular uniform, helicoidally uniform (with step, radius and speed of traveling constant) or rectilinear uniform.
In each one of these types the magnitude of the charge speed is constant during the motion, while the type of trajectory depends on the orientation of the starting velocity. In addition, the examination of the phenomena leads to the conclusion that there is a force \( \vec{F} \) acting perpendicular to the velocity vector \( \vec{v} \): \( \forall \vec{v} \in \mathbb{R}^3 \quad \vec{F} \perp \vec{v} \) at every point of motion. Moreover, because of the constant step of the helix, we deduce that the force acts only in the plane perpendicular to the axis of the helix, and the parallel velocity component to the helix remains constant. Also, depending on the motion of the charge, we see that the helix axis is parallel to the axis of the uniform circular motion and coincident with the direction of the rectilinear uniform one. These results allow to individuate a constant entity built up from the relevant vectors dynamically describing the system: \( \vec{B} \times \frac{\vec{P}}{q} \). A simple reasoning shows that this vector is essentially equivalent to the magnetic field vector. Rotating the system around the \( x \) axes (Figure 3), the magnetic field vector and a general position vector transform in the same way.

![Figure 3: Rotation of a simple electromagnetic system](image)

If we consider instead a reflection respect to the \( xy \) plane the two vectors transform in different ways.

![Figure 4: Symmetry transformation of a simple electromagnetic system](image)

So, in these examples, the magnetic vector transforms “normally” under the rotations of the system, but not under reflections.

**Introducing pseudovector to high school students**

In low level course physics, concerning three dimensional space, vectors are defined as abstract objects that are represented and characterized by: magnitude, direction, versus and (eventually) an application point. As concern operation with vectors, in particular, two external products are defined: the scalar product and the vector product. The first one associates to the two vectors a scalar quantity and the vector product associates to the two vectors a third vector (not belonging to
the initial vector space) and having perpendicular direction to the plane that carries the two starting
vectors and its verse is established using the right-hand rule.

More formally, a vector \( \mathbf{v} \) is the element of a vector space, i.e. an object that can be added or
subtracted to similar items and multiplied by the scalar. Given an \( n \)-dimensional Euclidean space
and a set of \( n \) linearly independent vectors \( \{ \mathbf{u}_1, \ldots, \mathbf{u}_n \} \), any vector of the space can be written as a
linear combination of these \( n \) linearly independent vectors:
\[
\mathbf{v} = a_1 \mathbf{u}_1 + \cdots + a_n \mathbf{u}_n.
\]
The \( \{ \mathbf{u}_i \} \) set is called a complete base for the vector space considered and every vector of the space
may be represent as the \( n \)-tuple of the real numbers \( \{ a_i \} \) so \( \mathbf{v} \equiv (a_1, \ldots, a_n) \).

From the other side, geometric transformation as rotation and symmetry rotation are formally
functions that map between two vector spaces (or - as in our case- from one vector space to itself)
and, if we limit our analysis to the case of a linear transformations, they preserve the properties of the
linear combinations of vectors. In particular linear geometric transformations can also be seen as
a base change in the vector space.

If we use a different complete base, for example \( \{ \mathbf{w}_i \} \), we would have \( \mathbf{v}_{\mathbf{w}} = b_1 \mathbf{w}_1 + \cdots + b_n \mathbf{w}_n \) so
\( \mathbf{v} = (b_1, \ldots, b_n) \).

In particular:
\[
\mathbf{v}_{\mathbf{u}} = b_1 \sum_{i=1}^{n} c_{1i} \mathbf{u}_i + \cdots + b_n \sum_{i=1}^{n} c_{ni} \mathbf{u}_i = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ji} b_j \mathbf{u}_i = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ji} \mathbf{w}_w = \begin{pmatrix}
    c_{11} & \cdots & c_{1n} \\
    \vdots & \ddots & \vdots \\
    c_{n1} & \cdots & c_{nn}
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    \vdots \\
    b_n
\end{pmatrix}
= \mathbf{v}_{\mathbf{w}}.
\]

So linear geometric transformations can also be seen as a base change in the vector space. In
particular, they can be represented by a matrix in the case of a transformation that maps from a
space in itself.

In the case that the two bases \( \{ \mathbf{w}_i \} \) and \( \{ \mathbf{u}_i \} \) are orthogonal, the \( \{ c_{ij} \} \) matrix is also orthogonal; i.e. it
is a square matrix with real entries whose columns (and rows) are orthogonal unit vectors. In
particular the determinant of this type of matrix is equal to \( \pm 1 \).

A reflection respect to a particular plane is a transformation of the vector space in itself
(endofunction and isomorphism) that maps every single point of the vector space in one and only
one point of the same space (bijective) without altering the distance between two starting points and
the two reflex points (isometric).

So is possible to define a linear application \( f \), representable as a matrix, that connect to each point \( P \)
of the space to his transformed and result for reflection \( f(\mathbf{P}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{pmatrix}
(\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}) = \begin{pmatrix} P_x \\ P_y \\ -P_z \end{pmatrix} \) and
for the considered rotation \( f(\mathbf{P}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{pmatrix}
(\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}) = \begin{pmatrix} -P_x \\ P_y \\ -P_z \end{pmatrix} \).

We notice that the first matrix as determinant equal to \( -1 \) and the second one has determinant \( \pm 1 \).
Experimentally we observe the strange behavior in the case of reflection, and in particular,
considering other type of transformation, can be show that for all transformation that has a
representative matrix having determinant equal to \( \pm 1 \), the magnetic field vector transform as a
‘normal vector’, for the other (that have determinant equal to \( -1 \)) the transformation rules are
different. In particular, concerning the reflection respect to the \( xy \) plane, experimentally:

\[
\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \rightarrow \begin{pmatrix} -B_x \\ -B_y \\ B_z \end{pmatrix}
\]
while if we apply the “standard” rule:

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} B_x \\ B_y \\ -B_z \end{pmatrix}.
\]
Since \( \mathbf{B} \) behaves differently from \( \mathbf{p} \) under particular transformation, they must be two different formal entities and this difference is highlighted in these particular context.

We go now to investigate the formal nature of \( B \). Considering a coil carrying an uniform electric current oriented in any way in space (relative to a orthonormal reference system \( xyz \)), from the Biot-Savart law the magnetic field vector is defined as:

\[
\mathbf{B} = \frac{\mu_0 I}{4\pi r^2} \int \frac{d\mathbf{l} \times \hat{r}}{r^2}
\]

That in the case of the magnetic field generated in the center of the coil, can be rewritten as:

\[
\mathbf{B} = \frac{\mu_0 I}{4\pi r^2} \int d\mathbf{l} \times \hat{r} = \frac{\mu_0 I}{2\pi r^2} (d\mathbf{l} \times \hat{r}) - \frac{\mu_0 I}{2\pi r^2} (d\mathbf{l} \times \hat{r})
\]

Posing \( k = \frac{\mu_0 I}{2\pi r^2} \), we obtain \( \mathbf{B} = k (d\mathbf{l} \times \hat{r}) \).

To discover the nature of \( B \) we must then go to investigate the nature of the vector product \( d\mathbf{l} \times \hat{r} \).

The vector product is usually defined as an application that maps from \( \mathbb{R}^3 \times \mathbb{R}^3 \) in \( \mathbb{R}^3 \) so at two vectors will be associated a third vector that does not belong in the starting space.

In particular the components of \( \mathbf{B} \) are of the type \( B_x = k(l_y r_z - l_z r_y) \) with cyclic permutation of the index. As can be seen the \( x \) component of \( \mathbf{B} \) depends only on the \( y \) and \( z \) components of the vectors \( \mathbf{r} \) and \( \mathbf{l} \). To emphasize this fact we can propose a change of notation: \( B_y = B_x \) (and similar).

Calculating all possible \( B_{ij} \) values and grouping them into a matrix we obtain:

\[
B = \begin{pmatrix}
B_{xx} & B_{xy} & B_{xz} \\
B_{yx} & B_{yy} & B_{yz} \\
B_{zx} & B_{yz} & B_{zz}
\end{pmatrix} = \begin{pmatrix}
0 & B_x & -B_y \\
-B_x & 0 & B_z \\
B_y & -B_z & 0
\end{pmatrix}
\]

This new notation allows to rewrite the magnetic field as an antisymmetric matrix with zero trace, that is uniquely associated to a set of three numbers \( \{B_x, B_y, B_z\} \) that, in the three dimensional space, with an abuse of notation are usually graphically represented as a vector \( \mathbf{B} \) having components

\[
\begin{pmatrix}
B_x \\
B_y \\
B_z
\end{pmatrix}
\]

The advantage given by this new representation is strictly connected to the rule of transformation. In fact matrix under basic transformation, transform according to a rule that differs from the one used for the vectors. In particular they transform following the law \( \mathbf{B}' = ^T S B S \); where \( \mathbf{B}' \) is the transformed of \( \mathbf{B} \) and \( S \) is the usual transformation matrix for vectors and \( ^T S \) is the transposed matrix of the matrix \( S \). So, in the case of reflection,

\[
B' = ^T S B S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix} \begin{pmatrix}
0 & B_x & -B_y \\
-B_x & 0 & B_z \\
B_y & -B_z & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix} = \begin{pmatrix}
0 & B_z & B_y \\
-B_z & 0 & -B_x \\
B_y & -B_x & 0
\end{pmatrix}
\]

and in the case of rotation

\[
B' = ^T S B S = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix} \begin{pmatrix}
0 & B_x & -B_y \\
-B_x & 0 & B_z \\
B_y & -B_z & 0
\end{pmatrix} \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix} = \begin{pmatrix}
0 & B_z & -B_y \\
-B_z & 0 & -B_x \\
B_y & -B_x & 0
\end{pmatrix}
\]

That are both consistent with the experimental exploration.

In the end, to find an agreement between experiment and theory, the magnetic field cannot be represented by a vector, but by a 3x3 matrix. Formally this is expressed by saying that the magnetic field is a tensor of order two.
Conclusions
In the path we proposed, starting from experimental exploration of phenomena, the students face a situation in which the “standard” representation of the magnetic field fails and, through the use of the formal description provide by mathematic, they review the definition of the formal entity. In this way the experimental framework allows students to stress the difference between vector and pseudovector (i.e. polar and axial vector) an the formal approach applied to the specific situations allow them to bridge one of the main highlighted gap in the student studies, providing them a formal description in which they face the ‘real’ formal nature of the magnetic field.

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References