Lecture 1: Survey of High Energy Phenomena

1.1 Introduction

What is high energy astrophysics anyway?

- Any phenomena occuring beyond the solar system that involves energies $E \gtrsim 1 \text{ keV}$ (where $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$ is the energy gained by an electron in a potential drop of 1 volt).
- This corresponds to temperatures $T \sim E/k \gtrsim 10^7 \text{ K}$ at which matter is fully ionised, so we are dealing with free particles in plasma.
- The plasma may be heated so efficiently that it reaches relativistic energies and does not have time to thermalise.
- The high energy particles include electrons, protons and other nuclei, as well as photons; most of our information comes from high energy photons at frequencies $\nu \sim E/h \gtrsim 10^{17} \text{ Hz}$ corresponding to X-rays and $\gamma$-rays.

What physics do we need to know?

- Some electromagnetic theory to understand radiation from fast moving charges and particle acceleration mechanisms.
- Astrophysical fluid dynamics, including some magnetohydrodynamics (MHD), to understand mass, momentum and energy transport in high energy environments and hydrodynamical acceleration mechanisms.
- Special and General Relativity to understand the relativistic limits of radiation processes and fluid dynamics.
1.2 Historical Perspective

The first experiments in radioactivity (c. 1900) revealed sources of ionisation that could not be accounted for by natural radioactivity. In 1912, Victor Hess made a manned balloon ascent and showed that the ionisation of air increases with altitude. This was the discovery of cosmic radiation, or cosmic rays. Hess was awarded a Nobel prize in Physics in 1936 for his discovery.

Most of the cosmic rays detected at Earth’s surface are in fact cascade products of much higher energy cosmic rays from the top of the atmosphere. With the development of rockets and satellites in the 1960’s, the study of cosmic rays and more generally, high energy astrophysics, has entered a new era.

The cosmic ray spectrum

- range of energies spans $10^7 \text{ eV} \lesssim E \lesssim 10^{20} \text{ eV}$; above $10^9 \text{ eV}$, the spectra for protons, electrons and nuclei can be described by

\[
N(E)dE \propto E^{-(2.5-2.7)}dE
\]  

(1)

where $N(E)$ is the number of cosmic rays per unit area per unit time per unit energy per solid angle.

- small deviations from this power-law profile are detected at $\sim 10^{15} \text{ eV}$ (the “knee”) and $\sim 10^{18} \text{ eV}$ (the “ankle”); interpretation is that cosmic rays have different origins: $E \lesssim 10^{10} \text{ eV}$ are primarily solar cosmic rays; $10^{10} \text{ eV} \lesssim E \lesssim 10^{15} \text{ eV}$ are of Galactic origin (supernova remnants, X-ray binaries); $E \gtrsim 10^{15} \text{ eV}$ are probably extragalactic.

The origin of ultra high energy cosmic rays (UHECRs), with $E$ above the ankle, is still unknown; if they are extragalactic, this poses a serious challenge for viable acceleration mechanisms.
1.3 Sources of High Energy Particles

The solar system

- **The Sun** produces solar energetic particles (SEPs) via solar flares and coronal mass ejections (CMEs); solar flares produce electrons with $E \sim 10 \text{ keV}$, which subsequently emit X-rays; exceptionally energetic flares and CMEs can accelerate ions to $\sim 10 \text{ MeV}$, resulting in $\gamma$-rays; escaping particles are detected in the solar wind by spacecraft.

- **Earth’s magnetosphere** accelerates particles at the bow shock, in the auroral zones, in the tail and in the Van Allen radiation belts ($E \sim 10 \text{ keV} - 1 \text{ MeV}$);

- **Jupiter’s magnetosphere** produces $1 - 10 \text{ MeV}$ electrons and ions in a similar way to Earth’s magnetosphere, but much more intensely owing to the motion of Io through the magnetosphere, and on a much larger scale, owing to its sheer size.

Beyond the solar system

- Most stars are binaries; close interacting binaries, usually involving a compact primary, undergo mass transfer and **X-ray emission** by accelerated electrons

- **Supernova explosions** send an expanding shock wave into the interstellar medium, accelerating particles to MeV energies; the relativistic electrons emit radiation up to X-ray energies.

The Crab supernova remnant (SNR) in X-rays.

The SNR Cassiopeia A: composite false-colour image showing optical (white/yellow), infrared (red) and X-ray (green) emission.
The Galaxy

The Galactic Centre (GC) is believed to be a source of high energy particles; many SNRs in the vicinity of the GC are also seen in X-rays and \( \gamma \)-rays. New observations with the HESS (High Energy Stereoscopic System) telescope array have revealed that interstellar gas clouds in the Galactic plane also emit \( \gamma \)-rays as a result of cosmic ray interactions in the clouds. Whether the primary cosmic rays originate from the GC is still unclear (but see Ballantyne et al., Astrophys. J. Lett., 657, L13, 1 March 2007 for a recent update).

Extragalactic sources

Quasars, radio galaxies and other active galaxies all exhibit high energy phenomena, e.g. X-rays, \( \gamma \)-rays, relativistic jets
Galaxy clusters are also sources of diffuse X-ray emission.

The coma cluster exhibits diffuse and extensive X-ray emission, suggesting that the intracluster medium is filled with very hot ($\sim 10^8$ K) gas with a total mass comparable to the total mass of all the stars in all the galaxies in the cluster! Similar X-ray emission is seen in individual galaxies and is believed to play an important role in galaxy formation and evolution.
Lecture 2: Astrophysical Acceleration Processes I

Particle acceleration mechanisms can be classified as *dynamic*, *hydrodynamic* and *electromagnetic*, although there is no clear distinction between these because the dynamics of charged particles are ultimately governed by electromagnetic fields, which pervade the entire Universe. Nevertheless, in some cases it is clear that magnetic fields, whilst present, are not dynamically important. We will now consider a standard astrophysical particle acceleration process known as *Fermi acceleration* which can operate effectively in the absence of magnetic fields. Fermi made the first serious attempt at explaining the power law nature of the cosmic ray spectrum. He noted that if cosmic rays are injected steadily into a localised acceleration region where they gain energy at a rate that is proportional to their energy while at the same time their escape from the region is an energy-independent Poisson process, then the stationary particle distribution will always be a power law. Fermi also argued that the most efficient acceleration mechanism would be a stochastic one.

2.1 Some Kinetic Background Theory

High-energy astrophysical sources contain plasmas, which consist of charged particles. These charged particles can have a distribution of momenta $p$ and this distribution can be a function of 3D space $x$ and time $t$. The distribution function is normally given by the parameter

$$f(x, p, t) \quad \text{particle distribution function} \quad (1)$$

This is also sometimes referred to as a *phase space density* because $f$ gives the number of particles in the 6-dimensional phase space defined by $p$ and $x$ at a given $t$. The *number density* of particles is the number of particles per unit volume at a particular location $x$ and is defined by

$$n(x, t) = \int f(x, p, t) \, dv \quad \text{particle number density} \quad (2)$$

where $dv = dv_x dv_y dv_z$. Particle conservation requires that the rate of change of the number of particles per unit time per unit volume is equal to the flux of particles across the surface of the volume (in the absence of sources or sinks). This conservation law is given by
the following:
\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\partial}{\partial \mathbf{p}} \left( \frac{d\mathbf{p}}{dt} f \right) = 0
\]
Vlasov equation (3)

which is sometimes also referred to as the collisionless Boltzmann equation because Coulomb collisions are assumed to be negligible. If electrostatic interactions between particles are important, then inclusion of a Coulomb collision term gives the Boltzmann equation. Under steady-state conditions, the Boltzmann equation gives the following well-known solution to the particle distribution function:
\[
f(\mathbf{p}, \mathbf{x}) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{\left| \mathbf{p} - \langle \mathbf{p} \rangle \right|^2}{2mkT} \right)
\]
Maxwellian distribution (4)

This is the distribution function for particles in thermal equilibrium. The mean kinetic energy is \( \varepsilon = \frac{1}{2} kT \) per particle per degree of freedom.

In many powerful astrophysical sources, large amounts of energy appear to be dissipated into the ambient medium over a relatively short timescale. If the gas density is high, then Coulomb collisions between electrons and ions will efficiently thermalise the plasma. The dissipated energy ends up distributed equally amongst the particles, forming a Maxwellian distribution. This process is referred to as plasma heating.

In this course, we will focus on particle acceleration, rather than particle heating. This refers to energy dissipation processes in which a minority of particles end up with the majority of the available energy. This may occur when the dissipation mechanism is fast enough and the plasma diffuse enough that there is not enough time for thermalisation (via Coloumb collisions) to redistribute the energy equally amongst the particle population. The resulting particle energy distribution is nonthermal and is usually a power-law: \( n(\varepsilon) \propto \varepsilon^{-p} \). The cosmic ray spectrum provides direct observational evidence that efficient cosmic particle accelerators exist. The challenge is to find a specific acceleration mechanism that can explain the power law slope of this spectrum: \( p \approx 2.5 \).
2.2 Fermi Acceleration

In 1949, Enrico Fermi proposed the first serious acceleration mechanism in an astrophysical context. He proposed that galactic cosmic rays are accelerated in the interstellar medium as a result of many collisions with massive, magnetised clouds which act as a scattering center, like cosmic billiard balls (see movie at spacephysics.ucr.edu).

Consider a particle of momentum $\mathbf{p}$ and energy $\varepsilon$ that collides with a massive cloud moving with a random velocity $\mathbf{V}$. In the centre of momentum frame of the cloud, the component of the particle’s initial relativistic three-momentum parallel to the direction of $\mathbf{V}$ is

$$p'_{\parallel i} = \Gamma \left( p_{\parallel i} - \frac{\varepsilon_i V}{c^2} \right)$$

where $\Gamma = (1 - V^2/c^2)^{1/2}$ is the Lorentz factor of the cloud. In this frame, the collision is elastic, so the final parallel component of the particle’s 3-momentum is $p'_{\parallel f} = -p'_{\parallel i}$ and its final energy is equal to its initial energy:

$$\varepsilon'_f = \varepsilon'_i = \Gamma (\varepsilon_i - V p_{\parallel i})$$

Transforming back to the lab (observer) frame, the final energy of the particle is

$$\varepsilon_f = \Gamma (\varepsilon'_f + p'_{\parallel f} V) = \Gamma (\varepsilon'_i - p'_{\parallel i} V).$$

Substituting the expressions for $\varepsilon'_i$ and $p'_{\parallel i}$ given above then yields

$$\varepsilon_f = \Gamma^2 \left[ \left( 1 + \frac{V^2}{c^2} \right) \varepsilon_i - 2 (\mathbf{V} \cdot \mathbf{p}_i) \right]$$

writing $\mathbf{p}_i = \varepsilon_i \mathbf{v}_i/c^2$, the observed change in particle energy is thus

$$\Delta \varepsilon = \varepsilon_f - \varepsilon_i = 2\Gamma^2 \left( \frac{V^2}{c^2} - \frac{\mathbf{V} \cdot \mathbf{v}_i}{c^2} \right) \varepsilon_i$$

(5)

Note:

1. For head-on collisions, $\mathbf{V} \cdot \mathbf{v}_i < 0$ and the particle gains energy.

2. For overtaking collisions, $\mathbf{V} \cdot \mathbf{v}_i > 0$ and the particle loses energy.

So after many collisions, is there a net energy gain? Yes, because there is a higher rate of head-on collisions than overtaking collisions.
Consider \( N \) scattering centres (clouds) per unit volume with collisional cross section \( \sigma \). The rate of encounters for a given direction of \( \mathbf{V} \) (but with \( \Gamma \approx 1 \)) is

\[
R \approx N \sigma |\mathbf{v} - \mathbf{V}| \approx N \sigma v \left(1 - \frac{\mathbf{v} \cdot \mathbf{V}}{v^2}\right)
\]

which demonstrates that head-on encounters (\( \mathbf{V} \cdot \mathbf{v_i} < 0 \)) are indeed more frequent than overtaking encounters (\( \mathbf{V} \cdot \mathbf{v_i} > 0 \)). The average energy gain per unit time follows by averaging \( R \Delta \varepsilon \) over all possible directions of \( \mathbf{V} \):

\[
\langle \frac{d\varepsilon}{dt} \rangle \approx N \sigma v \langle \left(1 - \frac{\mathbf{v} \cdot \mathbf{V}}{v^2}\right) \Delta \varepsilon \rangle
\]

where \( \Delta \varepsilon \) is given by (5) (omitting the ‘i’ subscript). To calculate the average quantities, we write \( \mathbf{v} \cdot \mathbf{V} = vV \cos \theta \) and average over all \( 0 \leq \theta \leq \pi \), noting that \( \langle \cos^2 \theta \rangle = \frac{1}{3} \) and \( \langle \cos \theta \rangle = 0 \) for an isotropic velocity distribution. The result is

\[
\langle \frac{d\varepsilon}{dt} \rangle \approx \frac{8}{3} N \sigma v \frac{V^2}{c^2} \varepsilon
\]

and the mean energy gain per collision is

\[
\frac{\langle \Delta \varepsilon \rangle}{\varepsilon} \approx \frac{\langle d\varepsilon/\varepsilon \rangle}{\langle R \rangle} \approx \frac{8}{3} \frac{V^2}{c^2} 2\text{nd-order Fermi acceleration}
\]\n
This is Fermi’s famous initial result, demonstrating that a statistical nett gain in energy results from collisions of particles off scattering centers, with an average energy gain that is second order in \( V/c \). This is a type of stochastic acceleration: the average systematic energy gain of a particle results from many small, non-systematic energy changes.

Interaction of a cosmic ray with an interstellar cloud moving at velocity \( \mathbf{V} \). Although 2nd-order Fermi acceleration can be simply described in terms of billiard ball collisions, the scattering process is somewhat more involved. As the particle enters the cloud, it scatters off irregularities in the internal magnetic field. So the particle’s final energy and momentum are the result of many scatterings inside the cloud.
2.3 The Spectrum Due to 2nd-Order Fermi Acceleration

The result (6), viz. $\langle d\varepsilon/dt \rangle = \alpha \varepsilon$, where $\alpha = \frac{8}{3} N \sigma v^2 \varepsilon / c^2$ implies that $\varepsilon(t) = \varepsilon_0 \exp(\alpha t)$, so the characteristic timescale of the acceleration process is $t_{acc} \sim \alpha^{-1}$. The particles can also escape from the region where the acceleration takes place. Suppose this occurs on a timescale $t_{esc}$. The flow of particle energy under the influence of stochastic Fermi acceleration can be described by a diffusion equation (formally derived from a Fokker-Planck approximation to the Vlasov equation, because the fractional energy-momentum changes in a single collision are small). Let $dn = n(\varepsilon, t) d\varepsilon$ be the number density of particles with energy in the range $(\varepsilon, \varepsilon + d\varepsilon)$. Then the evolution of the particle population can be described by

$$\frac{dn(\varepsilon, t)}{dt} + \frac{\partial}{\partial \varepsilon} \left[ (\frac{d\varepsilon}{dt}) n(\varepsilon, t) - \frac{\partial}{\partial \varepsilon} (Dn(\varepsilon, t)) \right] \approx - \frac{n}{t_{esc}} + Q(\varepsilon, t) \quad (8)$$

The term in square brackets on the LHS is the mean particle energy flux; it is the nett difference between the rates of mean energy gain and energy diffusion (where $D$ is an energy diffusion coefficient). The last term on the RHS of (8) is a source term describing injection of particles. If we neglect energy diffusion and the source term $Q$, and consider a steady-state solution ($dn/dt = 0$), then we find

$$- \frac{d}{d\varepsilon} (\alpha \varepsilon n) - \frac{n}{t_{esc}} \approx 0$$

Differentiating and rearranging gives

$$\frac{dn(\varepsilon)}{d\varepsilon} \approx - \left( 1 + \frac{1}{\alpha t_{esc}} \right) \frac{n}{\varepsilon}$$

and thus, $n(\varepsilon) \propto \varepsilon^{-p}$, which is a power-law particle distribution with spectral index $p = 1 + (\alpha t_{esc})^{-1}$. Although this was the reason why stochastic Fermi acceleration was initially a very promising mechanism for astrophysical sources, observations indicate that $p \simeq 2.5$ over a broad range of different sources. This means that the combination of the parameters $\alpha$ and $t_{esc}$ must be very fine-tuned, which is unlikely. Other effects have been considered, such as stochastic acceleration by MHD turbulence and plasma waves in general, but even so, a more favourable acceleration mechanism, particularly for ultra-relativistic particles, is Fermi's 1st-order acceleration which can occur naturally at astrophysical shocks.
Lecture 3: Acceleration Processes II

In the previous Lecture, we found that the scattering process underlying Fermi acceleration predicts a net particle energy change per scattering event

\[
\Delta \varepsilon = 2 \left( \frac{V^2}{c^2} - \frac{Vv \cos \theta}{c^2} \right)
\]

viz. eqn. (5) in Lec. 2 (for scatterers moving nonrelativistically). If the scatterers are moving in random directions, we found that the energy gain averaged over all directions is 2nd-order in \(V^2/c^2\) (c.f. eqn. (7) in Lec. 2). If, on the other hand, we could construct a special geometry such that the scattering occurs preferentially head-on (i.e. \(\cos \theta < 1\) always), then we should find a nett systematic energy increase \(\mathcal{O}(V/c)\) which reflects the rms change of energy.

Consider, for example, a unidirectional stream of idealised scattering centres moving towards an elastically reflecting wall. An incoming particle gains energy every time it completes a cycle of reflecting off the wall into the stream and scattering back towards the wall. This forms the basis of particle acceleration near an astrophysical shock.

3.1 Some Background: Physics of Shock Waves

Shock waves are ubiquitous in space and astrophysical plasmas. They describe disturbances in a plasma that propagate at a speed exceeding the sound speed for that medium. This means that sound waves cannot transmit information about the disturbance fast enough to plasma ahead of the shock. There is a discontinuity in the properties of the plasma ahead of the shock and behind it and the processes that occur across the shockfront are irreversible. The change in plasma properties are best described with a fluid dynamics approach.

Kepler’s supernova remnant is just one of many examples of shock waves in astrophysics. This shock wave is expanding at \(2000 \text{ km s}^{-1}\). The shock-ionised interstellar plasma emits radiation across almost the entire electromagnetic spectrum. The X-rays are attributed to shock-accelerated particles.
Consider a planar (1D) shock wave propagating with velocity $U$ through a stationary plasma. In the undisturbed region ahead of the shock (i.e. “upstream”), the plasma is at rest. The “downstream” region swept up by the shock is moving supersonically (but not as fast as $U$). In the rest frame of the shock, the upstream fluid moves towards the shock with velocity $v_1 = |U|$, pressure $p_1$, temperature $T_1$ and mass density $\rho_1$. The downstream fluid moves away from the shock with velocity $v_2$ and its pressure, temperature and density are $p_2$, $T_2$ and $\rho_2$, respectively (see figure).

(a) A shock wave propagating at supersonic speed $U$ through a stationary plasma.

(b) The fluid flow through the shock front in the rest frame of the shock.

The physical properties of the upstream and downstream fluid are related by the equations for conservation of mass, momentum and energy across the discontinuity (shock). For fluxes across a stationary boundary, these conservation relations are:

(i) **continuity equation:**
\[ \nabla \cdot (\rho \mathbf{v}) = 0 \implies \rho_1 v_1 = \rho_2 v_2 \]  

(ii) **momentum equation:**
\[ \nabla \cdot (\rho \mathbf{v}) \mathbf{v} = -\nabla p \implies p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \]

(iii) **energy equation:** this is equivalent to Bernoulli’s equation in fluid dynamics which defines streamlines of constant energy flux
\[ \nabla \cdot \left( \left( \frac{1}{2} v^2 + w \right) \rho \mathbf{v} \right) = 0 \implies \left( \frac{1}{2} v_1^2 + w_1 \right) \rho_1 v_1 = \left( \frac{1}{2} v_2^2 + w_2 \right) \rho_2 v_2 \]

Here, $w = \varepsilon_{\text{int}} + p/\rho$ is the specific enthalpy and $\varepsilon_{\text{int}}$ is the internal energy. For an ideal gas, the internal energy and pressure are related via an adiabatic index $\gamma$, with $p/\rho = (\gamma - 1)\varepsilon_{\text{int}}$ – e.g. for a monatomic ideal gas, $\gamma = 5/3$ and $p/\rho = \frac{2}{3} \varepsilon_{\text{int}} = kT/\mu m_p$. 


We now have 3 equations and 6 variables: $\rho_1, \rho_2, v_1, v_2, p_1, p_2$. We seek expressions for the downstream quantities (subscript ‘2’) in terms of the upstream quantities (subscript ‘1’). First we introduce the following definitions for the upstream fluid:

\[ r = \frac{\rho_2}{\rho_1} \quad \text{compression ratio} \]

\[ c_1 = \left( \frac{\gamma p_1}{\rho_1} \right)^{1/2} \quad \text{sound speed} \]

\[ M_1 = \frac{v_1}{c_1} \quad \text{Mach number} \]

Then we solve the conservation equations simultaneously. After some algebra, we obtain the following shock jump conditions:

\[ r = \frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\gamma + 1}{\gamma - 1 + 2/M_1^2}, \quad p_2 = \frac{2 \gamma M_1^2 - (\gamma - 1)}{(\gamma + 1)} \quad \text{Mach number} \]

Note that in the limit of very strong shocks ($M_1 \gg 1$), $r \to (\gamma + 1)/(\gamma - 1)$, so the compression converges to a finite value (e.g. $r = 4$ for $\gamma = 5/3$). But the pressure goes as $M_1^2$, so shocks can heat gas to arbitrarily high temperatures. In the fluid description, however, this is plasma heating rather than particle acceleration, which we look at next.

### 3.2 Fermi Acceleration at a Shock

In the late 1970’s, several researchers independently worked out that 1st-order Fermi acceleration of particles to high energies can occur at strong astrophysical shocks. In this mechanism, particles gain energy as a result of repeated scatterings across a shockfront.

For a strong shock in a fully ionised gas, we have $v_2 \approx \frac{1}{4}v_1$ and $v_1 = |U|$ in the rest frame of the shock. Now consider a fast particle in the upstream region. In the rest frame of the upstream fluid, the shock is advancing at speed $U$ and the downstream fluid is advancing at speed $\frac{3}{4}U$. The particle will scatter and spatial diffusion will bring it to the shock (because the fluid converges there). When the particle crosses the shock, it undergoes head-on collisions with the scattering centres in the advancing downstream flow. Now in the rest frame of the downstream flow, the upstream flow is advancing at a speed $\frac{3}{4}U$, so when the particle crosses the shock again, it incurs another energy gain. If the shock is subrelativistic ($U \ll c$) and the particle is relativistic, the fractional energy gain is $\Delta \varepsilon/\varepsilon \sim \left( \frac{U}{c} \right) \cos \theta$. 
Fractional energy gain:

The energy gain per complete cycle of shock crossings depends on the angle \( \theta \) between a particle’s momentum and the shock normal. Consider a relativistic particle moving across a strong, subrelativistic shock from downstream to upstream. Let us denote the restframes of the downstream and upstream flows with ‘+’ and ‘−’, respectively (see figure).

In the restframe of the upstream flow (where the shock is moving at speed \( U \) and the downstream flow is moving at speed \( V = \frac{3}{4} U \)), the particle’s initial energy upon entering is

\[
\epsilon^−_i = \epsilon^+_i \left( 1 + \frac{V}{c} \cos \theta_1 \right)
\]

where \( \theta_1 \) is the angle between the particle’s initial momentum and the shock normal. The particle then collides with a scattering centre in the upstream flow and re-crosses the shock at an angle \( \theta_2 \). Its final energy in the restframe of the upstream flow is

\[
\epsilon^−_f = \epsilon^+_f \left( 1 + \frac{V}{c} \cos \theta_2 \right)
\]

Since the scattering is elastic in this reference frame, \( \epsilon^−_f = \epsilon^+_i \) and so

\[
\frac{\epsilon^+_f}{\epsilon^+_i} = \frac{1 + (V/c) \cos \theta_1}{1 + (V/c) \cos \theta_2}
\]

Rearranging gives

\[
\frac{\Delta \epsilon^+}{\epsilon^+} = \frac{V}{c} \left( \cos \theta_1 - \cos \theta_2 \right)
\]
To calculate the average energy gain per cycle, we need to know that the probability of a particle crossing the shock at an angle between $\theta$ and $\theta + d\theta$ is $P(\theta) \propto \cos \theta \sin \theta d\theta$. This is because the number of particles arriving with $\theta$ in this range is proportional to $\sin \theta d\theta$, while the rate at which they arrive per unit time is proportional to $\cos \theta$. Using $\mu = \cos \theta$ and integrating over all $0 \leq \theta \leq \pi/2$ for head-on collisions over a complete cycle gives:

$$\frac{\langle \Delta \varepsilon \rangle}{\varepsilon} = \frac{V}{c} \int_0^1 d\mu_1 \int_0^1 d\mu_2 \mu_1 \mu_2 (\mu_1 - \mu_2)$$

$$\Rightarrow \frac{\langle \Delta \varepsilon \rangle}{\varepsilon} = \frac{4V}{3c} \text{ 1st order Fermi acceleration}$$

Thus, the average energy gain is 1st order in $V/c$. This Fermi mechanism is sometimes referred to as diffusive shock acceleration because it relies on the diffusion of the particles towards the shockfront.

Energy spectrum:

The energy spectrum for 1st-order Fermi acceleration can be calculated if we let $\varepsilon = \beta \varepsilon_0$ be the average energy of a particle after one cycle and let $P$ be the probability that the particle remains in the accelerating region after that cycle, i.e. $\beta = 1 + \frac{4V}{3c}$. Then after $k$ collisions, there are $n = n_0 P^k$ particles with energies $\varepsilon = \varepsilon_0 \beta^k$. Eliminating $k$ gives

$$\frac{n}{n_0} = \left( \frac{\varepsilon}{\varepsilon_0} \right)^{\ln P / \ln \beta}$$

The number of particles of energy $\varepsilon$ within the range $d\varepsilon$ is then

$$n(\varepsilon) d\varepsilon \propto \varepsilon^{-1 + \ln P / \ln \beta} d\varepsilon$$

which is a power law. To quantitatively determine the exponent in the power law, we need to find a steady-state solution to the particle continuity equation in the diffusion approximation, as we did with 2nd-order Fermi. This time, however, we cannot neglect the diffusion term.
Lecture 4: Acceleration Processes III

4.1 The Spectrum Due to 1st-Order Fermi Acceleration

In diffusive shock acceleration, all particles incident upon the shock must gain energy by varying amounts that depend on the number of times they cross the shock. It can be argued, qualitatively, that the steady-state distribution functions of the accelerated particles must necessarily be a featureless power law because there is no obvious momentum scale in the process. Only in the nonlinear regime, when the particle momenta are sufficiently high to cause a back reaction on the background plasma, does the particle rest mass become a crucial scale in the process. This limit is not considered here.

To calculate an explicit power law spectral index, we refer to the diffusion equation for particles (which is a Fokker-Planck equation in the limit of small differential changes in momenta, as is relevant here). Unlike the calculation for stochastic Fermi acceleration, however, we must include the effects of spatial diffusion, which is crucial to the diffusive shock acceleration mechanism.

Consider a plane shock located at a spatial coordinate $x = 0$. In the shock restframe, the fluid moves from upstream to downstream and we define this as the direction of increasing $x$. We want to calculate the steady state particle momentum distribution $f(p)$. In addition to the spatial diffusion resulting from scatterings, particles can also be convected (advected) with the fluid flow across the shockfront to the downstream flow. So terms involving spatial diffusion and convection (advection) need to be retained in the overall momentum diffusion equation, which can be written as:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x} \right) = 0$$  \hspace{1cm} \text{convection–diffusion equation} \tag{1}

Here, $v$ is the fluid speed and $D$ is the spatial diffusion coefficient. We can expand the last term and simplify, ignoring the term $\partial / \partial p (\partial v / \partial x)$ that describes the back reaction of the high-energy particles colliding with the scattering centres in the fluid. We get the following:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x} \right) = \frac{1}{3} p \frac{\partial v}{\partial x} \frac{\partial f}{\partial p}$$  \hspace{1cm} \tag{2}
The 2nd term on the LHS describes changes in the distribution function as particles convect along with the fluid in the downstream direction. The term involving $D$ describes how $f$ changes with location as a result of spatial diffusion associated with scattering. The term on the RHS describes changes in the particle distribution function resulting directly from changes in the particle momenta. This is the term associated with particle acceleration. To calculate a steady state solution for $f(p)$, we set $\partial f/\partial t = 0$. We then also set the acceleration term to zero and find a general solution that satisfies the balance between convection and diffusion. This general solution is then modified by considering the acceleration term and requiring that the final solution be continuous across the shock boundary.

First balancing convection and diffusion only, we have

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x} \right)$$

and we also write

$$v = \begin{cases} v_- , & x < 0 \text{ (upstream)} \\ v_+ , & x > 0 \text{ (downstream)} \end{cases}$$

This differential equation has the following general solution in the upstream region, where convection opposes diffusion:

$$f(x, p) = f_-(p) + [f_0(p) - f_-(p)] \exp \left[ \int_0^x \frac{v_- \, dx}{D(x, p)} \right] , \quad x < 0$$

with $f_-(p) = f(-\infty, p)$ and $f_0(p) = f(0, p)$. It is not possible to balance diffusion against convection behind the shock and the only possible steady state solution has $f$ spatially constant:

$$f(x, p) = f_0(p) , \quad x > 0$$

Our solutions satisfy equation (3) for the balance between convection and diffusion. But we really want a solution to

$$v \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x} \right) = \frac{1}{3} p \frac{\partial v}{\partial x} \frac{\partial f}{\partial p}$$

where the RHS term is the acceleration term. Note that it involves $\partial v/\partial x$, whereas our solution above contains a constant $v_-$ in the upstream region. So we may try to find a solution by letting $v$ vary spatially. Performing the derivatives and substituting into (6), we find the
requirement for a continuous momentum distribution across the shock:

\[
- D \frac{\partial f}{\partial x} - \frac{1}{3} v_p \frac{\partial f}{\partial p} \bigg|_{0+}^{0-} = 0
\]  

(7)

The two solutions (4) and (5) can now be joined at the shock using (7) to obtain a differential equation for the transmitted distribution function \( f_+(p) \):

\[
\frac{df}{d \ln p} = \frac{3r}{r-1} (f_+ - f_-)
\]

which has a solution

\[
f_+(p) = q p^{-q} \int_0^p dp' f_-(p') p'^{(q-1)}
\]

(8)

where

\[
q = \frac{3r}{r-1} \quad \text{and} \quad r = \frac{v_1}{v_2}
\]

This solution indicates that the downstream distribution \( f_-(p) \) acts as an injection spectrum. If we insert a monoenergetic spectrum, i.e. \( f_-(p) \propto \delta(p - p_0) \), into (8), we get a power law at \( p > p_0 \) of the form \( f_+(p) \propto p^{-q} \).

The energy spectrum for an ideal, monatomic gas and a strong shock is

\[
n(\varepsilon) \propto p^2 f(p) \propto \varepsilon^{-2}
\]

(9)

Unlike stochastic Fermi acceleration, this result is not sensitive to assumptions about escape losses. Furthermore, different assumptions about the transport equation produce essentially the same spectrum, so the result is robust. One problem, however, is that diffusive shock acceleration is more efficient for ions than for electrons, yet it is the radiation emitted by accelerated electrons that we observe and the inferred electron energies are high. Some form of pre-acceleration of electrons is therefore required. Also, it is not entirely clear whether diffusive shock acceleration can account for the highest energy cosmic rays, which are believed to be accelerated at the shocks associated with supernova explosions.
4.2 MHD Shocks

Magnetic fields can change the nature of shocks and can change the behaviour of incoming particles and hence, their rate of acceleration via the Fermi mechanism. Consider again the restframe of a planar shock. Suppose the upstream region contains a magnetic field $B_1$ that makes an angle $\phi_1$ with the shock normal. In the downstream region, the field is $B_2$ and makes an angle $\phi_2$ with the shock normal. If we define the shock normal as being in the $x$-direction and the other coordinate as the $z$-direction, then the magnetic field will have components that are normal and tangential to the shock. The fluid velocity in the downstream region can also have a tangential component in this case. We now determine how the shock jump conditions are modified by the presence of the magnetic field.

The field has no effect on the continuity equation, but does introduce extra terms in the momentum and energy equations. The momentum equation becomes

$$\nabla \cdot (\rho \mathbf{v}) \mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

(10)

where $\mathbf{J}$ is the current density. In the MHD limit, the average changes in the fields occur sufficiently slowly over the time and length scales of interest that the magnetic field behaves as part of the fluid. In particular, Ampere’s law in the MHD limit is $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ and using a vector identity, we get

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left( \frac{B^2}{2\mu_0} \right) + (\mathbf{B} \cdot \nabla) \frac{\mathbf{B}}{\mu_0}$$

Now we can separate the momentum equation into normal and tangential components:

$$\frac{\partial}{\partial x} \left[ \frac{1}{2} \rho v_x^2 + p + \frac{B_z^2}{2\mu_0} \right] = 0$$
$$\frac{\partial}{\partial x} \left[ \frac{1}{2} \rho v_x v_z - \frac{B_x B_z}{\mu_0} \right] = 0$$

(11)

The energy equation now has a contribution from the Poynting flux:

$$\nabla \cdot \left[ \left( \frac{1}{2} \rho v^2 + \frac{\gamma}{\gamma - 1} p \right) \mathbf{v} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right] = 0$$

(12)

We can use Ohm’s law

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Ohm’s law

for MHD in the infinitely conducting limit to get $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ so that

$$\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = \frac{1}{\mu_0} \left[ B^2 v - (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \right]$$
The energy conservation relation then reads
\[
\frac{\partial}{\partial x} \left[ \left( \frac{1}{2} \rho v^2 + \frac{\gamma}{\gamma - 1} p \right) v_x + \frac{B^2}{\mu_0} v_x - \frac{\mathbf{v} \cdot \mathbf{B}}{\mu_0} B_x \right]
\]
(13)

In addition, we have Maxwell’s relations, which require
\[
\frac{\partial B_x}{\partial x} = 0, \quad \frac{\partial}{\partial x} (v_x B_z - B_x v_z) = 0
\]

These boundary conditions now need to be combined. We define the following parameters:
\[
v_{1A} = \frac{B^2}{\rho_1 \mu_0} \text{ Alfvén velocity}
\]
\[
M_A = \frac{v_1}{v_{1A}} \text{ Alfvén Mach number}
\]
(14)

The change in the magnetic field across the shock is then given by
\[
B_{2x} = B_{1x}, \quad B_{2z} = B_{1z} \left[ \frac{r (M_A^2 - \cos^2 \phi_1)}{M_A^2 - r \cos^2 \phi_1} \right]
\]
(15)

Combined with the other jump conditions derived in the unmagnetised case, these are referred to as the Rankine-Hugoniot jump conditions. A magnetised shock is referred to as parallel when \( \phi_1 \approx 0 \) (i.e. \( \mathbf{B}_1 \) parallel to shock normal \( \hat{x} \), so \( B_z = 0 \)) and perpendicular when \( \phi_1 \approx \pi/2 \). Thus, there is no change in the magnetic field for a parallel shock, whereas for a perpendicular shock, the magnetic field increases across the shock.

The energy of a charged particle near an MHD shock is not modified by the magnetic field. However, the field controls the trajectory of the charged particle and can thus either enhance or diminish the Fermi acceleration process. Consider a perpendicular shock, for example, and suppose the particle energy is sufficiently large that its gyroradius is larger than the shock thickness (usually the case). As the particle gyrates around \( \mathbf{B} \approx \hat{B} \hat{z} \), it crosses the shock many times. The overall efficiency of Fermi acceleration is thus enhanced as a result of this higher probability of approaching the shockfront.
4.3 Direct Particle Acceleration by Electric Fields

The dynamics of charged particles are governed by electromagnetic fields, which pervade the entire Universe. The equation of motion of a particle of charge $q$, momentum $p = \gamma m v$ and Lorentz factor $\gamma = (1 - v^2/c^2)^{1/2}$ in a magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ is

$$\frac{dp}{dt} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (16)$$

In most (but not all) astrophysical situations, static E-fields cannot be sustained because ionised plasmas are very highly electrically conducting and the charged particles move freely to short out any component of $\mathbf{E}$ parallel to $\mathbf{B}$. Perpendicular to $\mathbf{B}$, particle motion is restricted by a B-field to a circular motion with gyrofrequency

$$\Omega = \frac{|q| B}{\gamma m} \quad (17)$$

The sense of gyration is right-handed for negative charges ($q/|q| = -1$).

The gyroradius is $R = v_\perp/\Omega = p_\perp/(|q| B)$, where $v_\perp = v \sin \alpha$ is the velocity component perpendicular to $\mathbf{B}$ and $\alpha$ is called the pitch angle. Combined with the parallel velocity component $v_\parallel = v \cos \alpha$, the net motion of a charged particle in a magnetic field is a spiralling motion.

From (16), the work done on a charged particle is $W = \mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = q \mathbf{v} \cdot \mathbf{E}$. Thus, since magnetic fields do no work and static electric fields cannot be sustained, acceleration of charged particles to high energies can only be attributed to the time-varying E-field induced by a time-varying B-field, viz. Faraday’s law: $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$.

Although diffusive shock acceleration appears to be an efficient and natural particle acceleration mechanism, whether it can account for the highest energy cosmic rays is still an open question. Let’s see what the requirements are for electric field acceleration. We can do the following order-of-magnitude calculation: $E_L \sim \frac{B}{L/c} \Rightarrow E \sim Bc$ where $L$ is a characteristic length scale over which the field varies. The total energy that can be given to a particle is

$$\varepsilon = \gamma mc^2 \sim \int qE dl \sim qBcL$$

We find $\varepsilon \sim 5 J \sim 10^{19} \text{ eV}$ for $B \sim 10^6 \text{ T}$ and $L \sim 100 \text{ km}$. These physical parameters are precisely what we find in neutron stars.