PROJECTILE SOLUTIONS ON MINKOWSKI DIAGRAM

Nagy Péter, *Kecskemét College Faculty of Mechanical Engineering and Automation*
Tasnádi Péter, *Eötvös Loránd University (ELTE) Faculty of Science*

Abstract

Physics is difficult; theory of relativity is very far even from the university students. Therefore it is very important to motivate students to understand the basic ideas of theoretical physics for example the principles of theory of relativity. In this paper, a problem (and its solution) connected to the popular novel "Lord of Rings" is presented which leads toward some questions of special relativity. Generalizing the problem it is also shown, that one dimensional problems of the theory of special relativity can be solved by geometrical constructions using the concept of Minkowski space-time.

1. Introduction

The construction of a Minkowski space time diagram highly supports the understanding of the principal ideas of the theory of special relativity. It gives the possibility of visualization and qualitative demonstration of the ideas which are so strange for the gumption. Numerical solution of the problems on the Minkowski diagram is not easy due to the distortion of the measure originating from the hyperbolic geometry of the space-time. The scaling of the coordinate axis of the moving system is determined by a hyperbola; therefore it is difficult to read the numerical value of the lengths and times involved into the problems. It will be shown, that through the comparison of the invariant scalars of the Euclidean geometry and those of the Minkowski geometry a scaling factor can be derived which gives exactly the distortion of the measures, so the solution of the problems can be carried out with the help of a ruler. These connections of the theory of special relativity are simple consequences of the basic postulates of the theory, but from everyday aspects they are strange and hardly interpretable results. As it often happens good illustrations make the understanding of the theoretical results easier and hereby the problem-solving too. For the illustration of the theory of relativity a brilliant geometrical diagram was constructed by Minkowski. Minkowski diagrams show two or more space–time coordinate systems in the same figure. Its main purpose is to allow for the space and time coordinates *x* and *t* used by one observer to read off immediately the corresponding *x′* and *t′* used by the other and vice versa. However, in text books the Minkowski diagrams are generally used for qualitative illustration of the connection between the coordinates measured by different observers. In the following a projectile method will be shown for the quantitative application of the Minkowski diagram.

In usual form of a one-dimension Minkowski diagram the *x* distance is displayed on the vertical axis and the time (more exactly the $c \cdot t$ quantity) on the horizontal axis. Draw the $(ct)^2 - x^2 = 1$ hyperbola in this diagram. This hyperbola goes through the point (0,1) However, the quantity of $(ct)^2 - x^2$ is invariant, therefore $(ct')^2 - x'^2 = 1$ at the point, where the hyperbola intersects the t'-axis of reference frame *S'* (where *x' = 0*). It means that *ct' = 1 m*, so we get the geometrical representation of the time unit in S' (Taylor and Wheeler, 1992).

All these conclusions are very clear, the problem merely remained weather how that hyperbola can be drawn exactly?! Thinking over this question, it can be conclude that the procedure presented above only gives a way for the determination of the time unit. But it is not enough to give the complete scale in *S'*. Oddly enough, text-books disregard this problem although without the exact scaling Minkowski diagram is inapplicable to transform the coordinates of an event from *S* to *S'*. It is worth mentioning that one of the authors of this presentation looked through hundreds of internet links searching this procedure and has not found it. However, it is apparent that the scales of the axes of reference frame *S'* are determined by the relative velocity *S'* consequently a scale-factor *η* must exist, which gives the connection between the scales of the *S'* and *S* systems.

2. The scale-factor

The scale-factor may be deduced more on the basis of the scale-deformation of the Minkowski diagram which is really the consequence of the hyperbolic geometry of space-time.

Since the $\left[(c\Delta t)^2 - \Delta x^2\right]$metric of hyperbolic geometry is scaled through the Euclidean metric $\left[(c\Delta t)^2 + \Delta x^2\right]$:

$\eta^2 \cdot \left[(c\Delta t)^2 - \Delta x^2\right] = \left[(c\Delta t)^2 + \Delta x^2\right]$,

consequently:

$$\eta = \sqrt{\frac{(c\Delta t)^2 + \Delta x^2}{(c\Delta t)^2 - \Delta x^2}} = \sqrt{\frac{1+\left(\Delta x/c\Delta t\right)^2}{1-\left(\Delta x/c\Delta t\right)^2}} = \sqrt{\frac{1+\left(v/c\right)^2}{1-\left(v/c\right)^2}} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}\,.$$

3. A motivating example

Nowadays one of the cinema-hits is Tolkien's first-rate yarn, the Lord of the Rings. (The map of Middle Earth is also applied in a problem of the famous book of Weinberg (1972).) The scene of the story is Middle-earth which is a strange world, perhaps one of its most important properties – which isn't mentioned by Tolkien, being linguist and not physicist! – is that the speed of light is altogether 100 km/h. Merry, one of the chief characters of the story receives a wonderful horse from the king of Rohan which witchcraft-horse is magically speedy: it can gallop at a speed of 75 km/h (three-quarters of the speed of light). In the final battle – in which the Fellowship of the Ring fought with the Dark Lord below the walls of Minas Tirith – Merry galloping beside the standing Witch-king, chopped his head (let's name this with not-canonized simplicity event *A*).

(a) Draw up the Minkowski diagram of *S* reference frame fixed to the Witch-king, and apply on both axes [100 km] = [45 mm] scale unit! Represent on this diagram scale-properly the axes of the reference frame *S'* fixed to the galloping Merry, if we know that event *A* happened according to the clock of *S* (Witch-king) at $7/9$ o'clock, yet according to the clock of *S'* (Merry) at $100/85$ o'clock!
(b) Not to long after the death of the Witch-king (event *A*), according to Merry's clock exactly at 0,9 hour Minas Morgul (the Witch-king's Tower) was collapsed. Let this collapse be event *B,* (the local coordinate of the tower is in system *S* x_B= -50 km.) How much time elapsed according to the Witch-king's clock between the Witch-king's death and the breakdown of tower?
(c) Merry heading for the battle was compelled to gallop across the Deads' Valley (Paths of the Dead). He right at the entrance of the valley set going his own watch, according to which he passed over the valley exactly during 10 minutes. How long is the valley?
(d) What kind of colour did Merry see the Witch-king's blood-red shield when he galloped toward the King?

Solutions

(a) $\beta = \frac{3}{4}$; $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{4}{\sqrt{7}} = 1{,}512$; $\eta = \sqrt{\frac{1+\beta^2}{1-\beta^2}} = \sqrt{\frac{25}{7}} = 1{,}89$; L'=1,89*45 mm ≈ 85 mm is the scale unit. Thus the slope of the time-axis of *S'* system is $\beta = 3/4$ and passes trough the point *A* with coordinates $(x=0;\ t=7/9)$ (see in Fig. 1.a).

The origin of *S'* system can be determined from the fact that the time-coordinate of point *A* according to *S'* is $100/85$ hour, thus the origin of the coordinate system is situating backward from point *A* on the time axis by this measure. In this case this is a distance of 100 mm on a the straight line going through point *A*. The *x'* axis cross the *y'* axis at the origin and has a slope of $\frac{1}{\beta} = \frac{4}{3}$

Knowing the axes of the two system every further question can be answered.

(b) Consider on the Minkowski diagram (see Fig. 1.b) the intersection point *B* of $x_B = -50\ \mathrm{km}$ line (shown by dashed line),which is parallel with $c \cdot t$-axis and line of $t'_B = t'_A + 0{,}9\ \mathrm{hour}$ (parallel with x'-axis) and project it to $c \cdot t$-axis, the time-difference of the *A* and *B* events can be measured on the system *S* with a ruler. We get $\Delta t = (10mm/45mm)\cdot 1hour = \underline{0{,}22\ \mathrm{hour}}$.

(c) Consider the time coordinates in *S'* of the points of the Minkowski diagram representing the event when Merry entered into the valley and passed over it and denote their position at the $c \cdot t'$-axis *VE* and *VV* respectively. The time-coordinate of point *VE* is $t' = 0$, and that of *VV* is $t' = 10\ \mathrm{min} = (1/6)\ \mathrm{hour} \cdot 85\ \mathrm{mm} = 14{,}2\ \mathrm{mm}$,projecting this line to the **x**-axis (the thickened vertical section on Fig. 1.b) $\underline{\Delta x = 8{,}5\ \mathrm{mm} = (8{,}5mm/45mm) \cdot 100\ \mathrm{km} = 18{,}9\ \mathrm{km}}$ can be obtained.

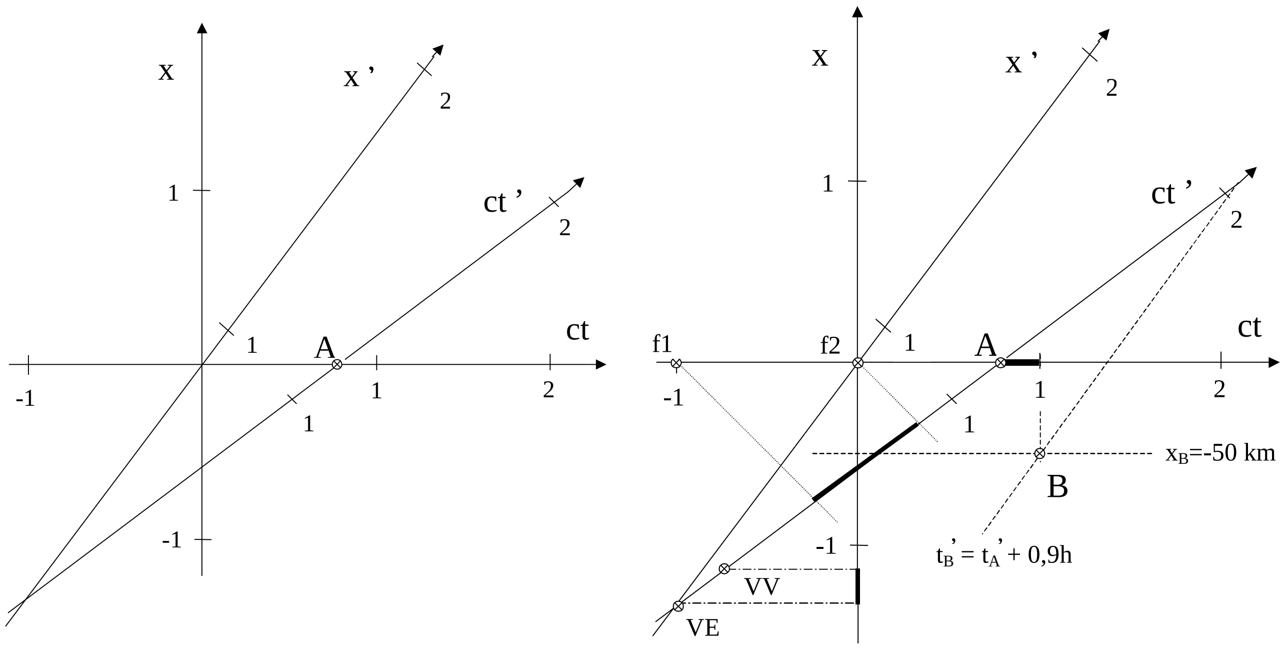

Figure 1.a and Figure 1.b

(d) According to the relativistic formula of Doppler-effect the deformation of wave-length

$\lambda' = \lambda\sqrt{\dfrac{1-\beta}{1+\beta}} = 0{,}378\lambda$. Consider in the system *S* the period of light investigated as unit. Indicate on the time axis a line segment of unit length, and project it by world lines to the time axes of *S'*. The length of the projection is the period of light in *S'*. Since the ratio of the length of a time interval measured on scales of two coordinate systems is independent from their lengths, the ratio gained above can be used in the case of arbitrary interval. Therefore the frequency requested can be obtained by multiplication of the frequency measured in *S* by the factor we determined above.

4. Testing the method

This method was successfully applied at the *Kecskemét College* by Peter Nagy in teaching special relativity in an introductory modern physics course.

To illustrate this we have compared the results of two groups of students who solved the same complex problem taken from special relativity. This problem can be solved either by the use of the basic formulas of special relativity (Lorentz transformation, length contraction, time dilatation etc.) or without any formula only with the application of the Minkowski diagram. Members of both groups were familiar with the basic formulas, but only one of the groups was practiced in the constructions on the Minkowski diagram (Group A). Table 1. shows that Group A reached significantly better achievement than the other one (Group B). It is worth noting that in question d) where the use of the Minkowski diagram was not relevant the two groups reached similar score.

The problem

The Hogwarts School of Witchcraft and Wizardry is a strange world, perhaps one of its most important properties is that the speed of light is altogether 100 m/s.

At the present time there is a quidditch-match, the title bout between the Gryffinder House and Slytherin House. The umpire of match Madame Hooch is levitating over the centre of the circle-shape quidditchground, when the Golden Snitch (one little ball) rush past her (exactly in the

direction to the box of teachers), in the wake of the ball Harry Potter races only with half second handicap according to the clock of Madam Hooch. The speed of Golden Snitch is 60 m/s (according to Madam Hooch), and the speed of the Firebolt (Harry's broom) is 80 m/s, so Harry will catch up the Golden Snitch soon, let this be event *A.*
Let be *S* reference frame fixed to Madame Hooch and the reference frame *S'* fixed to Harry Potter. Let set going Madame Hooch's watch in the moment when the Golden Snitch rush past her, and let start Harry's watch in the moment when he catch up the ball.
(a) Not to long after event *A* – according to Harry's clock exactly 1.5 second later – professor Snake sitting in the box of teachers will have a seizure (let note event *B*). In the *S* reference frame $x_B = x_A + 250m$. How much time elapsed between the event *A* and event *B* according to the Madame Hooch's clock? *(6 points)*
(b) Everybody think if professor Snake had a seizure, because Harry Potter catch up the Golden Snitch. May be any causal connection between the two events? *(2 points)*
(c) According to his clock Harry passed over the radius of circle-shape quidditchground (from the centre to the margin of circle) exactly during 2.5 seconds. How long is the radius in effect (in the *S* reference frame)? *(6 points)*
(d) How much is the speed of the Golden Snitch according to the reference frame *S'* fixed to Harry Potter? *(6 points)*

Results

Table 1.

	Group A (84 students) using Minkowski diagram						Group B (67 students) without using Minkowski diagram				
	a (6)	b (2)	c (6)	d (6)	Σ (20)		a (6)	b (2)	c (6)	d (6)	Σ (20)
mean	3,50	1,06	4,10	3,64	12,30		2,57	0,60	3,31	3,61	10,09
deviation	2,30	0,92	1,86	2,24	6,00		2,36	0,84	2,22	2,29	5,12

In the table the average points of the two groups with the deviations are shown.

5. Summary

Knowing the scaling factor it is possible to solve exactly all the one dimensional problems (length contraction, time dilatation, relativistic velocity addition and arbitrary dynamical question) belonging to the frame of the special relativity by using only geometrical constructions on the Minkowski diagram without the application of any further algebraic equation. Practically speaking the method presented in the paper encodes the Lorentz transformation into the structure of the Minkowski diagram and therefore the diagram involves all the consequences of the Lorentz transformation.
These involves twofold advantages: on the one hand it makes easier the realization of the special theory of relativity, on the other all problems can be solved with two entirely different methods (with formulas, and by geometry), so this offers the possibility of self-control for students.

References

Taylor E. and Wheeler J. A.(1992): *Spacetime Physics,* W. H. Freeman Publisher
http://www.rafimoor.com/english/SRE.htm
http://hubpages.com/hub/Using-the-Minkowski-Diagram-
Weinberg S (1972): *Gravitation and Cosmology*, J. Wiley, New York. USA. (p. 7)