DAMPING OF THE OSCILLATIONS AND EQUILIBRIUM OF A MASS-SPRING SYSTEM MECHANICAL MOTION DUE TO DIFFERENT FRICTION FORCES

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Abstract

Students generally have difficulty in understanding friction and its associated phenomena. High school and introductory university courses usually do not give the topic the attention it deserves and do not emphasize the crucial role of friction in establishing the mechanical equilibrium i.e. the end of a physical motion.
In this paper the physical situation of a periodical motion of a mass-spring system subjected to static and kinetic friction forces is proposed as an interesting case to be explored both experimentally and theoretically. A model which includes the effects of the static friction force to determine the final mass position is presented. A detailed comparison between theoretical and experimental results is discussed.

1. Introduction

Friction phenomena are everywhere around us allowing life as we know it. However at standard introductory university courses, and even in an higher formation level, not much attention is devoted to an in-depth study of mechanical motions in the presence of dissipative forces. Furthermore recent researches on student conceptions have highlighted several difficulties that students encounter in understanding friction [1, 2].
Purpose of the present work is to show how students can become familiar with a physical system presenting friction forces through experiments involving the periodic motion of a common mass-spring system out of its equilibrium position.
In an ideal non-dissipative mass-spring system the motion due to the spring restoring force alone is characterized by undamped periodic oscillations around a centre of motion, position where the resultant force on the mass is equal to zero. An external constant force acting on the mass does not affect the period of each oscillation, but simply shifts only the position of the centre of motion. Furthermore the centre of motion is also the only initial position, with zero velocity, of stable equilibrium.
Usually a spring-mass system in presence of a viscous friction force is discussed, i.e. damped oscillations are presented due to a friction force proportional to the velocity which yields an exponential amplitude decay. The physical departure from the ideal oscillatory model of the undamped periodic oscillations if a kinetic friction force is present is however generally less considered. Furthermore the role of static friction is often neglected.
Attention is here addressed to report a theoretical [3], [4], [5] and an experimental study of a spring-mass system oscillatory motion in the presence of a kinetic friction force and care will be reserved in evaluating the role of the static friction force in determining the final mass position. The experiments described in the subsequent paragraphs have been tested with high school students and teachers in a course for physics teacher education.
Theoretical and experimental results comparison is presented and discussed.
All ideas emerged in the so-called ‘Scientific degrees project’ involving high school students inside the environment of the physics university *Alessandro Volta* of Pavia for laboratory activities.

2. From static equilibrium to the damped oscillations: a theoretical approach

A mass-spring system, mass m, elastic constant k, oscillates constrained to an inclined-plane (θ being the inclination angle) in presence of both static and kinetic friction forces. Newton's law projected on a in-plane one-dimensional axis (oriented downward) can thus be written as:

$$m\ddot{x} = -k(x-\bar{x}) - sign(\dot{x})\mu_k mg\cos\theta + mg\sin\theta, \qquad (1)$$

where μ_k is the kinetic friction coefficient[1], $sign(\dot{x})$ denotes the sign function necessary for taking into account the dependence of the kinetic friction force on the direction of the velocity, $\bar{x}$ defines the rest position of the spring and mg is the modulus of the gravitational force.

While the modulus of the normal and the tangential components of the gravitational force are constant, (1) highlights the periodic switch in the direction of the dynamic friction force after each half period of motion $T/2 = \pi\sqrt{m/k}$.

The mass position where the spring restoring force balances the tangential component of the weight is $x_g = mg\sin\theta / k + \bar{x}$. Thus in absence of friction forces x_g represents the centre of oscillation and also the position with zero velocity of stable equilibrium.

By assuming $x_g = 0$ and indicating with $x_0 \neq 0$ the initial position of the mass ($\dot{x}_0 = 0$) the physical condition that makes the motion possible according to the previous dynamical law (1) is:

$$k\,|\,x_0\,| > \mu_s mg\cos\theta, \tag{2}$$

where μ_s denotes the static friction coefficient. The previous condition states the existence of a range of stable equilibrium positions, i.e. all the points $x_0 \in [-x_s, +x_s]$ with $x_s = \mu_s mg\cos\theta / k$.

If $|\,x_0\,| > x_s$ the oscillatory motion starts in agreement with (1). Suppose $x_0 > x_s > 0$.

In the first half period equation (1) has the form:

$$m\ddot{x} = -k(x - \bar{x}) + \mu_k mg\cos\theta + mg\sin\theta, \tag{3}$$

where both the magnitude and the direction of the sliding friction force are constant in time, the latter pointing downward, i.e. opposite to the upward direction of the velocity, so that its solution is:

$$x(t) = (x_0 - x_c)\cos(\omega t) + x_c, \tag{4}$$

with $\omega = 2\pi / T$ and $x_c = \mu_k mg\cos\theta / k$. It is worthwhile to note that due to the presence of the kinetic friction constant force, x_c is the position where the net force on the mass is zero i.e. it is the centre of motion of the first half oscillation.

The $\dot{x}(T/2) = 0$ position can be evaluated from (4) as:

$$x_1 = -x_0 + 2x_c. \tag{5}$$

Supposing $|\,x_1\,| > x_s$ in the next half period the particular solution of (1) changes due to the switch in the direction of the velocity and so of the sliding friction force. Thus the solution is now in the form:

$$x(t) = (x_1 + x_c)\cos(\omega t) - x_c, \tag{6}$$

being $-x_c$ the new oscillation centre, symmetrical with respect to x_g. The position after one period results:

$$x_2 = -x_1 - 2x_c = x_0 - 4x_c. \tag{7}$$

The general nth-amplitude at the nth-half period can thus be written as :

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[1] Few courses in basic physics associate Coulomb's name with the law of kinetic friction although he first provided in 1779 its treatment that has stood the test of time. His law is expressed in terms of two constant coefficients, one being of kinetic type μ_k, the other static μ_s, with the latter being larger than the former. Surprising to most, the magnitude of the friction force acting on a body sliding on a surface is at a first order independent of both the speed and the area of contact, depending only on the normal force acting on the touching surfaces.

$$x_n = (-1)^n (x_0 - 2nx_c) \quad if \;\; x_0 > x_s > 0 \,, \tag{8}$$

or

$$x_n = (-1)^n (x_0 + 2nx_c) \quad if \;\; x_0 < x_s < 0 \,. \tag{9}$$

Previous equations state that the amplitude damping is linear according to the rule $| x_n | - | x_{n-1} | = -2x_c$ after each half cycle. Furthermore from previous arguments the condition on the integer number n of half oscillations before the motion stops can be found. Considered $| x_0 | > x_s$ if the motion stops at x_n it has accomplished the nth-half oscillation ones. From the system of equations:

$$\begin{cases} | x_n | \leq x_s \\ | x_{n-1} | > x_s \,, \end{cases} \tag{10}$$

and according to (8) an (9), the motion stops at the first integer n satisfying:

$$\frac{1}{2}\frac{| x_0 | - x_s}{x_c} \leq n < \frac{1}{2}\frac{| x_0 | - x_s}{x_c} + 1. \tag{11}$$

It is possible to find the dependence of the final position x_n on the initial one x_0 (with $\dot{x}_0 = 0$) for a given spring-mass system. In Fig.1 the plot of x_n versus x_0 is reported. The plot is divided by vertical dashed lines in regions corresponding to different numbers of half oscillations made by the mass before it stops.

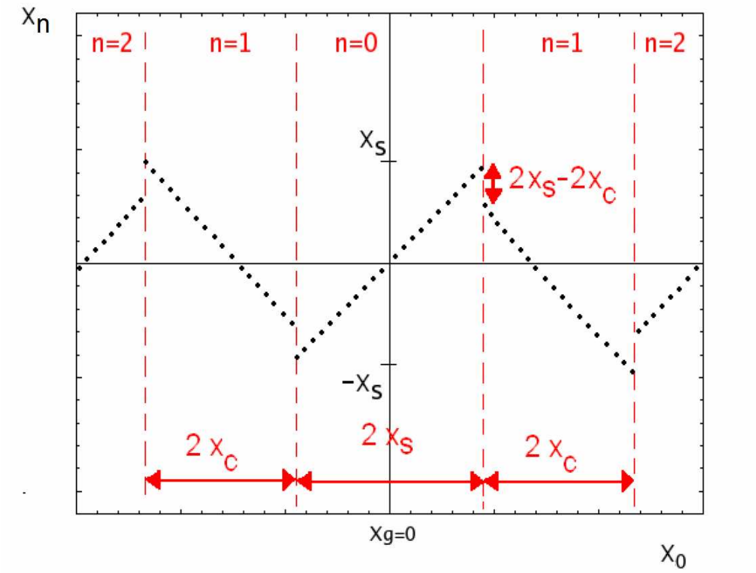

Figure 1: Representation of the final block position x_n for different initial positions x_0 of a damped harmonic oscillator in presence of both a static and a sliding friction force. Regions corresponding to different integer number n of half oscillations are highlighted by vertical dashed lines. The gap $2x_s - 2x_c$ between dotted lines together with the value of the x_c/x_s ratio allows an estimate of both the static and the dynamic friction coefficients. Furthermore the slope of the dotted lines is 1.

Due to the presence of a static friction force if $-x_s \leq x_0 \leq x_s$ the mass does not oscillate. Thus the central $2x_s$ wide region represents the static region of stable equilibrium where the final position x_n is coincident with x_0. The regions corresponding to $n = 1$ of width $2x_c$ represent the range of final positions of the block after one half oscillation. The gap $2x_s - 2x_c$ between two tilted dotted lines together with the value of the ratio:

$$\frac{x_c}{x_s} = \frac{\mu_k}{\mu_s}, \tag{12}$$

allows also an estimation of both the static and the dynamic friction coefficients for a given system. The slope of the dotted lines is 1.

3. System parameters: a measurement of the kinetic μ_k and of the static μ_s friction coefficients

The real mass-spring system under investigation consists of a block with four small plates made of plastic material placed beneath it. The block is free to oscillate on a wooden-inclined plane, coated with carpenter ant, since it is attached to a spring fixed at the top of the plane. Thus its oscillatory motion occurs in presence of both static and kinetic friction forces.
To measure the parameters of the block-spring system, i.e. the friction coefficients, two approaches are proposed. Since the direction of the sliding friction force depends on the direction of the velocity, the kinetic friction coefficient μ_k can be evaluated by measuring the different acceleration in the ascent a_a and the descent a_d motion. Indeed from Newton's equations projected on a one-dimensional axis (oriented downward) it is possible to write:

$$v_a < 0; \quad a_a = g\sin\theta + \mu_k g\cos\theta,$$
$$v_d > 0; \quad a_d = g\sin\theta - \mu_k g\cos\theta.$$

It is thus deduced:

$$\mu_k = \frac{a_a - a_d}{a_a + a_d}\tan\theta. \tag{13}$$

The block is thrown from the bottom to the top of the wooden-inclined plane and the upward and downward motions are registered by means of a motion sensor (connected to the computer through the PASCO interface and using the DATASTUDIO software) placed at the top of the plane, as represented in Fig.2 (left panel).

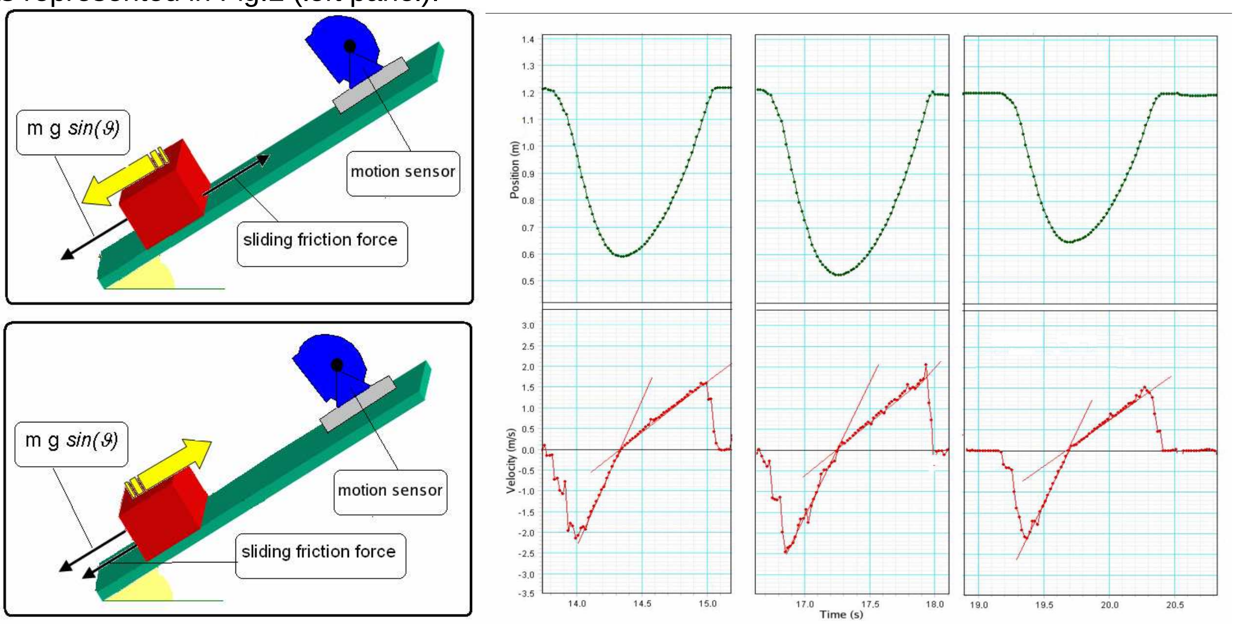

Figure 2: On the left the experimental approach to estimate the dynamic friction coefficient μ_k involved in the block-spring setup. On the right measurements of the position (upper part) and the velocity (downer part) of the block during the upward and the downward motion, got throwing the block from the bottom to the top of the inclined wooden-plane, recorded by means of a motion sensor. The sensor is placed at the top of the plane and computer interfaced.

Measurements of the position and the velocity versus time are reported in Fig.2 (right panel).The motion results in two uniformly accelerated motions (the upward and downward ones) with different acceleration. Thus from (13) the value for the kinetic friction coefficient μ_k has been obtained:

$$\mu_k = 0.25 \pm 0.03.$$

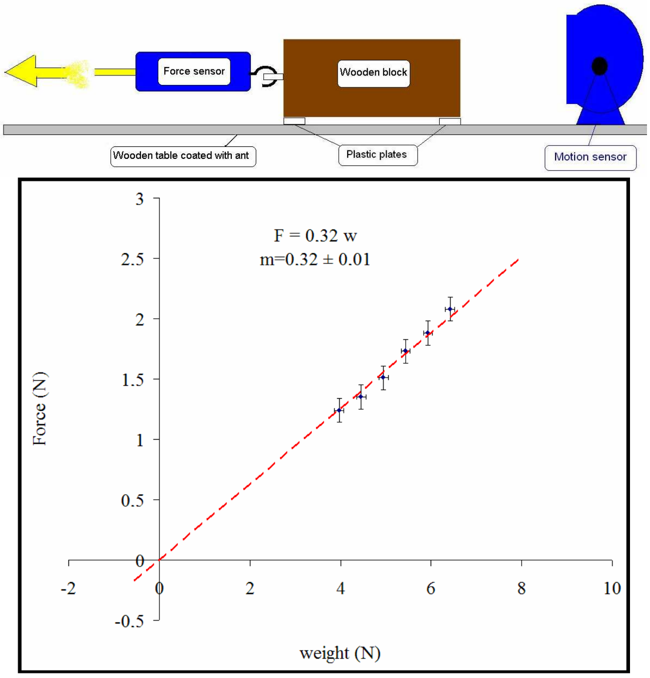

Figure 3: A measurement of the traction force threshold value needed to produce the block motion. The measurement has been done placing the wooden-plane horizontal. Both a force sensor attached to the block and a motion sensor are used. On the right the static friction coefficient μ_s *as estimated from the linearity between* $mg\cos\theta$ *and* F_s *defined as the threshold traction force value before the motion starts.*

In order to evaluate the static friction coefficient μ_s, the static friction force F_s defined as the threshold value of the traction force before the motion starts has been measured for different values of the normal force. To change the normal force known masses have been added on the top of the block. Both a force and a motion sensor are used, as reported in Fig.3 (left panel). Several values ($mg\cos\theta$, F_s) are collected as Fig.3 (right panel) shows. The linearity between the normal force $mg\cos\theta$ and the static friction force F_s allows to estimate the static friction coefficient. It turns out:

$$\mu_s = 0.32 \pm 0.01.$$

From previous results the μ_k/μ_s ratio for the block-spring system is:

$$\frac{\mu_k}{\mu_s} = 0.8 \pm 0.1. \qquad (14)$$

4. Damping due to the sliding friction force

A picture of the block-spring system with the friction parameters used in the analysis of the previous section is reported in Fig.4.

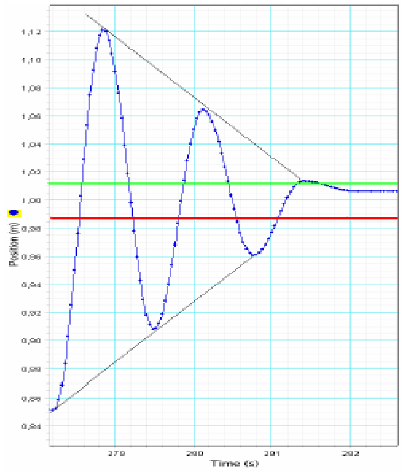

Figure 4: The experimental set up involving a periodic damped motion consists of a block-spring system placed on a wooden-inclined plane and a computer motion sensor at the bottom. Beneath the block four small plates of plastic material are placed.
Figure 5: Position versus time for a damped harmonic oscillator in presence of both a static and a kinetic friction force. Continuous black tilted lines drive the eye to follow the linear amplitude decay after a period of motion. Continuous horizontal green and red lines highlight the presence of two symmetrical centres for each half period depending on the direction of the motion, i.e. on the direction of the sliding friction force.

The tilted-angle chosen $\theta \sim 1.4$ rad allows the spring-block system to perform a series of oscillations so that the linear decay in amplitude can be appreciated. An initial non-equilibrium position x_0 (with $\dot{x}_0 = 0$) outside the static region is chosen and the position of the block during all the motion has been registered by the motion sensor placed at the bottom of the wooden-inclined plane.
In the graph position versus time shown in Fig.5 two centres of oscillation are evident symmetrical with respect to $x_g = 1m$. By assuming as in the theoretical model $x_g = 0$ their position are $\pm x_c \sim \pm 0.013m$. The amplitude decay agrees with (8) and (9), i.e. in each half cycle $|x_n| \sim |x_{n-1}| - 2x_c$.
Furthermore by choosing $x_0 \sim -0.15m$ as the initial position six half oscillations are obtained before the motion stops. The previous value corresponds, as derived from (11), to the first integer number belonging to the range:

$$5.1 \leq n < 6.1, \qquad (15)$$

where both (12) and (14) have been used to evaluate x_s.
The motion stops in the position given by (9) with $n = 6$, i.e. at $x_6 \sim 0.006m$, final position very close to the experimental one (see Fig.5).
To compare the plot of Fig.1 with the experimental values, several couples of initial and final position have been recorded by means of the motion sensor.
For these measurements the angle of the wooden-inclined plane has been reduced at $\theta \sim 1$rad to extend both the regions displayed in Fig.1 and to increase the number of experimental points.
Results are displayed in Fig.6.
From a qualitative point of view the agreement with Fig.1 is very good. Two discontinuous tilted lines appear denoting, as previously discussed, regions of x_0 values in which the block can accomplish respectively no oscillation (this is the static region of stable equilibrium positions) and one half-oscillation. It is also visible the range of x_0 values where the block can accomplish the second half oscillation. Furthermore the slope obtained by the linear fit of the data in the $n = 1$ region agrees with the theoretical expected value.
Particular measured values of x_c and x_s are:

$$x_s = 0.049 \pm 0.002\ m,$$
$$x_c = 0.043 \pm 0.002\ m. \qquad (16)$$

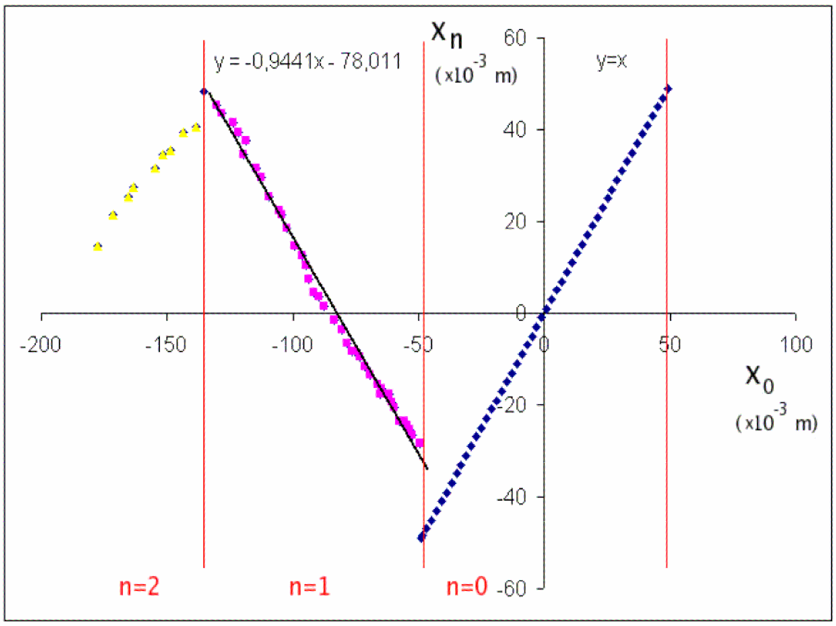

Figure 6: Final position x_n versus the initial position x_0 for the damped block-spring oscillator in presence of both static and sliding friction forces. Both positions have been recorded by means of the motion sensor placed at the bottom of the wooden-inclined plane. Regions corresponding to different integer number n of half oscillations are separated by vertical lines. Experimental points are placed on two discontinuous tilted lines as highlighted with a linear fit in the region of $n=1$; the slope is close to 1 as expected. Evaluated values of the width of the first two regions are $2x_s = 0.098 \pm 0.004m$ and $2x_c = 0.086 \pm 0.004m$.

From (12) the ratio x_c/x_s expresses the ratio between the friction coefficients belonging to the block-spring system. The value:

$$\frac{x_c}{x_s} = \frac{\mu_k}{\mu_s} = 0.9 \pm 0.1,$$

is found to be consistent with the one calculated in (14).

5. Conclusion

In this paper we have proposed an experimental approach to highlight the behavior of an harmonic oscillator in presence of both static and kinetic friction forces and a comparison with the theoretical model has been given. The problem of an oscillatory motion is, without any doubt, one of the main topics in physics, while a detailed study of its damping is not usual. By using motion and force sensors the presence of two centres of oscillation due to the periodic switch in the direction of the sliding friction force after each half period and the linear amplitude damping due to its magnitude have been highlighted also from an experimental point of view. Furthermore the role of the static friction force to arrest the oscillatory motion has been discussed and verified by experiments. Finally the quantitative analysis of the plot of the final position versus the initial position allowed a direct estimate of the ratio of the friction coefficients belonging to the block-spring system.

Note: More information on the use of the experiments with students and on the obtained results are reported in Onorato P., Mascoli D. and DeAmbrosis A. (2010) *Damped oscillations and equilibrium of a mass-spring system subjected to sliding friction forces: integrating experimental and theoretical analysis* accepted on Am. J. Phys.

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