FORTUNE WHEEL AS A TOOL FOR ILLUSTRATION THE BASIC PRINCIPLES OF STATISTICAL PHYSICS

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Abstract
A real-life situation is presented which gives an opportunity to illustrate the use one of the most important methods of statistical physics. The problem originates from an experience of Peter Nagy (one of the authors of the present paper). A fortune wheel will be studied and it will be shown that how the distribution of the prize labels on it can be determined to get a fair and exciting game. Motivating strength of the everyday life is used to reach better understanding of ideas of probability theory and statistical physics.

1. Motivation

On a nice Sunday afternoon of 2006 a long queue of impatient people were waiting for playing a game on a kind of roulette in a Hypermarket at Kecskemét (Hungary). The roulette wheel had 24 congruent sectors, 6 of them were labelled "NO PRIZE"; the others had labels of different PRIZE (Fig. 1.) On the basis of the geometrical ratio of sectors the chance of winning something seemed to be 75%. In spite of this, watching the games for a short time all the 50 trials occurred resulted in "NO PRIZE". Obviously, the wheel of fortune was manipulated.

Figure 1: the fortune-wheel in the shopping centre (one frame from video)

Naturally, a fair, not manipulated or influenced roulette wheel (Fig. 2.) should have such labels, which leads to geometric probability distribution. It means that the area of each sector is proportional to the probability of winning.

Figure 2: a real, fair fortune-wheel

2. The pedagogical benefit

The game with the false fortune wheel was a part of a monumental side-show and gambling game action in the Shopping Centre, so lots of students attended lectures on statistical physics courses of the University of the Town played some games with the wheel. Therefore they also observed the swindle, so they were also very curious to understand the background of the game, so this problem led to an interesting discussion in the classroom. Discussing the game two basic questions arisen: "How can the game be made attractive and exciting one?, How can it be made believable that the game is fair and the gamblers has real chance to win?"

The main benefit of the discussion was that students discern the expedience of the scientific ideas and the necessity of the exact formulation of the question.

3. Ideas and constructions

Students quickly understood that the core ideas of the problem are the definition of a "fair game" which involves the determination of the chance of winning. The requirement that the game should be exciting generated a lot of quarrel, but students agreed that it means the result of a single game must be unforeseen.

Nevertheless, they recognised that if somebody wants to give any exact formulation of the problem it is necessary to define the meaning of a fair game. Having a suitable definition for it the treatment of the problem of the game investigated needs the decision whether the game with the wheel is fair or not. To decide this it is necessary to construct a model for the fortune wheel which is working in a clear and understandable way.

According to the probability theory a game is a fair one if the gambler's expectation equals his stake. The gambler's expectation is obviously the amount he stands to win multiplied by the probability of his winning it. Generally the basic ideas of the gambling theory are unusual for students therefore a very simple example was discussed to illustrate the definition given above. The example was the following: If a gambler is offered 10 units each time he tosses a head with a true coin, and pays 5 units for each toss is this fair game? The gambler's expectation is 10 units multiplied by the probability of throwing a head, which is 1/2. His expectation equals exactly that he stakes on the game, so the game is fair. Generally the gambler's expectation equals with the probability weighted sum of the possible winnings.

4. A simple model of the fortune wheel and the game

Understanding the most important ideas it is possible to construct a model of the fortune wheel and formulate exactly the question connected with it. The model planned was the following.

The wheel had a revolving disk of 1 meter radius which has 100 sectors labelled with 11 different prizes (e.g. 0 €, 1 €, 2 €, …,10 € prizes). A spining pointer was applied to choose the prize.

Let the stake of a game with this wheel 2 €, and the play should also fulfil the following two requirements: (1) the average profit of the owner of the wheel (the casino) should be 0.04 € per game, (2) the game should be fair and as exciting as possible. It means that the result of a single game should be unforeseen. (Later it will be shown that entropy can be applied as a measure of excitement.)

Let us determine the number of sectors with different labels and their order on the roulette wheel (disk).

5. The solution

It will be shown, that the methods of statistical physics are suitable for the solution of the problem. Firstly, the terminology of the statistical physics is transferred to the fortune wheel which is now the system investigated (Reif, 1967).

The possible states of system are the labels on the circumference of the disk. According to the model there are $N = 100$ states. The relevant quantity which is to be examined is the value of prize on the label (It corresponds to the energy value of a state in statistical physics.) Possible prizes are quantified and have $m = 11$ different values:

$$\varepsilon = 1 \text{ €}, \qquad \varepsilon_i = i\,\varepsilon, \qquad i = 0, 1, …, 10.$$

Denote the number of state (sectors) with i-th label on the disk by N_i, so: $\sum_{i=0}^{10} N_i = N$.

The relative frequency of i-th label: $p_i = \dfrac{N_i}{N}$, and of course $\sum_{i=0}^{10} p_i = 1$.

According to requirement (1) the mean value of the prize is 2 €–0.04 €=1.96 €:

$$\overline{\varepsilon} = 1.96\in \quad \text{(the macroscopic condition)}.$$

There are a great number of distributions of labels that fulfils this macroscopic condition, for example a trivial distribution:

$$\{N_i\} = \{N_0 = 2,\ N_1 = 0,\ N_2 = 98,\ N_3 = 0,\ \ldots,\ N_{10} = 0\}.$$

6. *Entropy as a measure of excitement*

However, the simple distribution presented above is very transparent and do not satisfy requirement (2). In this case the game is not exciting, but it is very dull. Transform requirement (2) to the language of statistical physics (Dugdale, 2007). The terms entropy, uncertainty, and information are used more or less interchangeably, but from the perspective of probability, uncertainty is perhaps a more accurate term. Surprisal is the degree to which you are surprised to see a result. The uncertainty can therefore be thought of as a weighted average of the surprisal, so can be obtained the well-known Shannon-Boltzmann formula of entropy and information (see for example on: http://fourier.eng.hmc.edu/e161/lectures/compression/node4.html). It means that the distribution of labels should be as irregular as possible, that is *the entropy of the system should be maximal.*

So {p_i} distribution with maximum entropy should be determined of course the distribution must satisfy the two conditions which were already imposed.

$$\sum_{i=0}^{10} p_i - 1 = 0 \ : \text{condition of normality,}$$

$$\sum_{i=0}^{10} p_i \varepsilon_i - \overline{\varepsilon} = 0 \ : \text{macroscopic condition.}$$

Applying the Lagrange's multiplication method (Amit and Verbit, 1998):

$$\overline{S}(p_i) = -k \sum_i p_i \ln p_i = \max$$

$$\begin{cases} \Phi_0(p_i) = \sum_i p_i - 1 = 0 \\ \Phi_1(p_i) = \sum_i \varepsilon_i p_i - \overline{\varepsilon} = 0 \end{cases}.$$

The well known result can be obtained (Rasetti, 1986):

$$p_i = Ce^{-\beta \varepsilon_i}.$$

Since the relevant quantity is quantised:

$$p_i = p_0 q^i \text{, where: } p_0 = C \text{ , } q = e^{-\beta \varepsilon}.$$

The partition function is: $Z = \sum_i e^{-\beta \varepsilon_i} = \sum_i e^{-i\beta \varepsilon} = \sum_i q^i = \dfrac{1}{1-q}$.

Solving the conditional equations:

$$1 = \sum_i p_i = \sum_i p_0 q^i = p_0 \sum_i q^i = p_0 Z = \frac{p_0}{1-q},$$

$$\bar{\varepsilon} = \sum_i \varepsilon_i p_i = p_0 \sum_i \varepsilon_i q^i = -\frac{1}{Z} \sum_i (-\varepsilon_i) e^{-\beta \varepsilon_i} =$$

$$= -\frac{1}{Z} \frac{\partial \left(\sum_i e^{-\beta \varepsilon_i} \right)}{\partial \beta} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{\partial (\ln Z)}{\partial \beta} = \frac{\partial \left(\ln \frac{1}{Z} \right)}{\partial \beta} = \cdot$$

$$= \frac{\partial (\ln (1-q))}{\partial \beta} = \frac{\partial \left(\ln \left(1 - e^{-\beta \varepsilon} \right) \right)}{\partial \beta} = \frac{\varepsilon e^{-\beta \varepsilon}}{1 - e^{-\beta \varepsilon}} = \frac{\varepsilon q}{1-q}$$

The general solution: $p_0 = \dfrac{\varepsilon}{\varepsilon + \bar{\varepsilon}}$; $q = \dfrac{\bar{\varepsilon}}{\varepsilon + \bar{\varepsilon}}$.

The solution in our case:

$$N_i = N p_i = N p_0 q^i \text{ , where } \begin{cases} p_0 = \dfrac{\varepsilon}{\varepsilon + \bar{\varepsilon}} = \dfrac{1 \in}{1.96 \in + 1 \in} = \dfrac{1}{2.96} \\ q = \dfrac{\bar{\varepsilon}}{\varepsilon + \bar{\varepsilon}} = \dfrac{1.96 \in}{1.96 \in + 1 \in} = \dfrac{1.96}{2.96} \end{cases} .$$

So the distribution which was sought:

$$\begin{cases} N_0 = 33, \ N_1 = 22, \ N_2 = 15, \ N_3 = 10, \ N_4 = 7, \\ N_5 = 4, \ N_6 = 3, \ N_7 = 2, \ N_8 = 2, \ N_9 = 1, \ N_{10} = 1 \end{cases} .$$

Finally, Fig. 3 shows the fortune wheel made with sectors which corresponds the distribution obtained. The various prizes are marked with different colours. (The labels of the fortune wheel was generated by students (generally in excel)).
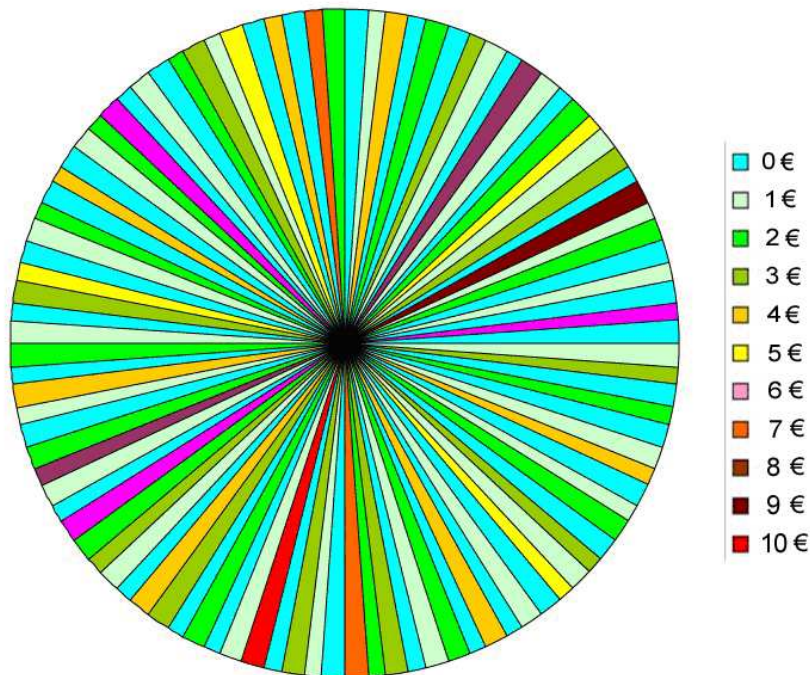
Figure 3: our fortune-wheel

This wheel does not cheat the player, it obviously realise the requirements of fair game. The distribution –diagram can be seen in Fig. 4.
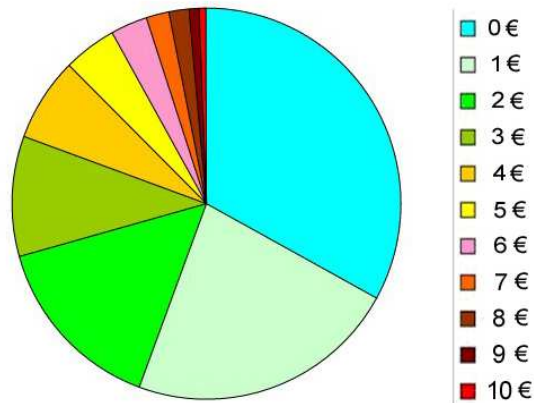

Figure 4: the distribution-diagram of our fortune-wheel

In contrast the estimated distribution –diagram of manipulated fortune wheel of the Hypermarket is shown in Fig.5. It is clear, that the manipulated wheel gave almost not any chance the gamblers for winning.
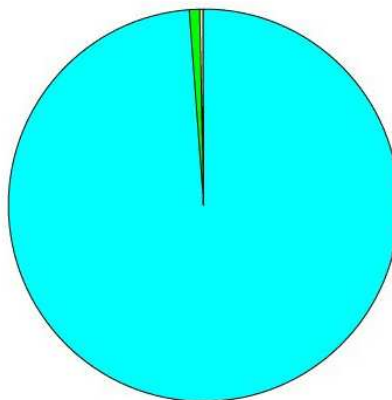

Figure 5: the estimated distribution of the manipulated wheel

7. Summary

Students can be motivated with solution of problems come from the real life. The interest of the students of the University of the Town was also piqued by the wheel. Methods and ideas of statistical physics can be transferred to the field of gambling. The discussion of the material was supported by interactive simulations and it was demonstrated with the help of pictures and videos too.(A version of the presentation with interactive illustrations can be downloaded from http://fiztan.uw.hu/fortunewheel/fw_present.zip.)

References

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