FFP12 2011
Udine 21-23 November 2011

GENERAL RELATIVISTIC QUANTUM THEORIES
Foundations, the Leptons Masses

Claudio Parmeggiani, *University of Milan (clparm@infinito.it*)

Abstract
The space-time is represented by a usual, four-dimensional differential manifold, X. Then it is assumed that at every point x of X, we have a Hilbert space $\mathcal{H}(x)$ and a quantum description (states, observables, probabilities, expectations) based on $\mathcal{H}(x)$. The Riemannian structure of X induces a connection on the fiber bundle $\mathcal{H}$ and this assumption has many relevant consequences: the theory is regularized; the interaction energy is a well defined self-adjoint operator; finally, applying the theory to electro-weak interactions, we can obtain a "specter" of leptons masses (electron, muon, tau).

Introduction. Summary
We shall discuss (the foundations of) a General Relativistic Quantum Theory: now the Hilbert space of the quantum description is replaced by a *fiber bundle* (here called *quantum bundle*) based on a four-dimensional differential manifold (the space-time); the typical fiber is a complex, infinite-dimensional Hilbert space; on every fiber there are states (vectors or rays), observables (operators), probabilities and expectations (Section 1),
The ordinary derivative (in Schrödinger equations) is replaced by a sort of *covariant derivative* on the quantum bundle. This derivative, at a space-time point, depends on the value of the space-time metric tensor (the gravitational field) at the same point: this is the *main hypothesis*. The standard quantum field theories are formally recovered, for a "constant" metric tensor (that is when the gravitational field goes to zero); but, in fact, even for an isolated elementary particle we have to consider the gravitational field of the particle itself (Section 2).
The quantum fields and the energy operator (the "Hamiltonian") are now well defined (eventually self-adjoint) operators; the Pauli-Jordan distributions are now true, differentiable functions (gravitation dependent, obviously); therefore the Theory is regularized: there are no more *divergences* (Section 3).
The presence of our "covariant" derivative not only regularizes the theory but, as a consequence, the (ratios of the) masses of the leptons are, by some supplementary assumptions, theoretically predictable. In fact, imposing the constancy of the sum of the space-time dependent *self mass* (logarithmically diverging, with zero gravity) and of a space-time dependent *counter term*, we arrive at a second order partial differential equation; the self values of this equation are (proportional to) the leptons masses (Sections 4 and 5).
Finally note that we have only to postulate the existence of three stable elementary leptons (electrons, positrons and electron neutrinos); the other ones (the μ and the τ particles, their neutrinos) and their masses are derived, so to speak, as excitations.

1. Fiber Bundles and Quantum Bundles
A *fiber bundle* (see, for example, (Bourbaki 2007) or (Lang 1985)) is a tuple $(\mathcal{F}, X, \pi, \mathbf{F})$ where $\mathcal{F}$ and X (the *base space*) are manifolds; π is a continuous surjection from $\mathcal{F}$ to X; the $\mathcal{F}(x) := \pi^{-1}(x)$ (the fibers at x, a point of X) are all isomorphic to the typical fiber **F**. A (cross) *section* is a function Ψ from X to $\mathcal{F}$ such that:

$$\pi(\Psi(x)) = x \tag{1}$$

Locally (in an open subset U of X) $\mathcal{F}(x)$ can be identified to F; hence, locally, the sections can be expressed as (ordinary) maps from U to **F**; in the following we shall generally employ *local expressions.*
A *quantum bundle* is defined as a tuple $(\mathcal{H}, X, \pi, \mathbf{H})$ where the base space X is a four-dimensional Riemannian manifold (**g** is its (*+1*, *–1*, *–1*, *–1*) metric tensor); the fibers at x, $\mathcal{H}(x)$, and the typical fiber, **H**, are infinite-dimensional, complex, isomorphic Hilbert spaces. The Schrödinger function is now a section of the quantum bundle, locally expressed by a map from X to **H**; the observables are

expressed by operator-valued maps, assigned on *X*. For example, the average value of the observable ξ, the "system" being in the state Ψ, can be (locally) written as:

$$(\Psi(x)|\xi(x)|\Psi(x))_{\mathbf{H}} / (\Psi(x)|\Psi(x))_{\mathbf{H}} \tag{2}$$

where the inner product is taken on **H**. Obviously in a Schrödinger picture the section ξ will be "constant" (in a sense make precise in subsection 2.2).

2. *Derivatives. Dynamics*

To relate quantum objects defined at different space-time points we need a way to connect them, at least when they are "infinitely near"; thence we shall introduce a notion of "covariant" derivative.

2.1. Covariant Derivatives

If Ψ is the local expression of a *section* of the Quantum Bundle and *u* is a 4-dimensional *vector* (space-like or time-like), the *covariant derivative* of Ψ, in the direction *u*, is defined as:

$$D_u\Psi(x) := \partial_u\Psi(x) + C_u(x).\Psi(x) \tag{3}$$

where ∂_u is the ordinary derivative (along *u*) and $C_u(x)$ is a space-time dependent operator on the Hilbert space **H**. And for an observable (an operator) ξ:

$$D_u\xi(x) = \partial_u\xi(x) + [C_u(x), \xi(x)] \tag{4}$$

If γ is a smooth path on *X*, parameterized by the real variable *s*, a section Ψ along γ is said *parallel* if:

$$D_u\Psi(\gamma(s)) = 0 \quad (u = \gamma'(s)) \tag{5}$$

2.2. Schrödinger Equation

The self-adjoint operator on **H** (*u* is again a 4-dimensional vector)

$$P_u(x) = P_0(x)\, u^0 + P_1(x)\, u^1 + P_2(x)\, u^2 + P_3(x)\, u^3 \tag{6}$$

is the (local expression of the) e*nergy-momentum* observable, at space-time point *x*; it is an energy operator if *u* is time-like, a momentum operator when *u* is space-like. Typically $P_u(x)$ can be decomposed as:

$$P_u(x) = P_u^{(\mathrm{FREE})}(x) + P_u^{(\mathrm{INT})}(x) \tag{7}$$

where $P_u^{(\mathrm{INT})}(x) = 0$, for a space-like *u*.
A section Ψ satisfies a *Schrödinger equation* (in a Schrödinger picture) if

$$D_u\Psi(x) = \partial_u\Psi(x) + C_u(x).\Psi(x) = -i\, P_u(x).\Psi(x) \tag{8a}$$

while, for *all* observables ξ:

$$D_u\xi(x) = \partial_u\xi(x) + [C_u(x), \xi(x)] = 0 \tag{8b}$$

Observe that, because $P_u(x)$ and $P_v(x)$ commute and $D_uP_v(x) = 0$ (for every couple of vectors, *u* and *v*), we have $D_uD_v\Psi(x) = D_vD_u\Psi(x)$.
In an *interaction picture*

$$D_u\Psi^{(\mathrm{INT})}(x) = -i\, P_u^{(\mathrm{INT})}(x).\Psi^{(\mathrm{INT})}(x) \tag{9a}$$

$$D_u\xi^{(\mathrm{INT})}(x) = i\, [P_u^{(\mathrm{FREE})}(x), \xi^{(\mathrm{INT})}(x)] \tag{9b}$$

In the following we shall always use the interaction picture.

2.3. Explicit expression of the Covariant Derivative

Having defined a *vacuum state* (at space-time point *x*), $\Phi_0(x)$, others states (and observables) can be generated by means of the *creation* and *annihilation* operators, $\mathbf{a}^*_{n,p}(x)$ and $\mathbf{a}_{n,p}(x)$ (see, for example, (Bogoliubov 1980) or (Weinberg 1995)). Here *n* is an index = *1, 2,* …, *N* and

$$p = (\varepsilon, \mathbf{p}), \quad \varepsilon = \sqrt{(\mathbf{p}^2 + M^2)} \tag{10}$$

ε, **p** and *M* are the particles energy, momentum and (bare, unobservable) mass.
Evidently we must have, for the vacuum, $D_u\Phi_0(x) = \partial_u\Phi_0(x) + C_u(x).\Phi_0(x) = 0$ (and also $\partial_u\Phi_0(x) = 0$). In our interaction picture

$$[P_u^{(FREE)}(x), \mathbf{a}_{n,p}(x)] = -(p.u)\ \mathbf{a}_{n,p}(x) \tag{11}$$

so: $D_u\mathbf{a}_{n,p}(x) = -i\ (p.u)\ \mathbf{a}_{n,p}(x)$. Afterward we shall assume (this is a *fundamental postulate*) that:

$$[C_u(x), \mathbf{a}_{n,p}(x)] = i\kappa\ c_u(x,p)\ \mathbf{a}_{n,p}(x) \tag{12}$$

where c_u is an *ordinary numerical function* dependent on *x* and *p* (but *not* on *n*) and, at least in a first approximation, that:

$$c_u(x,p) = -\sum_{\alpha\beta} \Gamma_u^{\alpha\beta}(x).\ p_\alpha p_\beta = \tfrac{1}{2} \sum_{\alpha\beta} \partial_u g^{\alpha\beta}(x).p_\alpha p_\beta \tag{13}$$

Here the $g^{\alpha\beta}$ are the "contravariant" components of the metric tensor **g**; the Γ are its Christoffel symbols; κ is a numerical, universal parameter. We are using absolute unit of measure, that is $h_{PLANK} = 2\pi$ and $M_{PLANK} = 1$.
Hence we are lead to the differential equation ($c(x,p) := \tfrac{1}{2} \sum_{\alpha\beta} g^{\alpha\beta}(x).p_\alpha p_\beta$)

$$i\ \partial_u\mathbf{a}_{n,p}(x) = ((p.u) + \kappa\ \partial_u c(x,p))\ \mathbf{a}_{n,p}(x) \tag{14}$$

which can be immediately integrated:

$$\mathbf{a}_{n,p}(x) = \exp(-ip(x-x_0)).\exp(-i\kappa(c(x,p) - c(x_0,p))).\mathbf{a}_{n,p}(x_0) \tag{15}$$

x_0 is a *fixed* space-time point. Evidently for a constant, "special relativistic" **g**, we shall recover the usual definitions, relatively to the free creation and annihilation operators (see again (Bogoliubov 1980) or (Weinberg 1995)).
For a "near constant" **g** (that is in a weak gravitational field approximation, see (Weinberg 1972))

$$c(x,p) = \tfrac{1}{2} M^2 + (M^{(GRAV)} / r(x))(\varepsilon^2 + \mathbf{p}^2) \tag{16}$$

$M = \sqrt{(\varepsilon^2 - \mathbf{p}^2)}$ is the bare mass; $M^{(GRAV)}$ is the gravitating mass; $r(x)$ is the distance from *x* to the source of the gravitational field.

2.4. Commutations and Anti-commutations Relations

At the same space-time point x_0

$$[\mathbf{a}_{m,p}(x_0), \mathbf{a}_{n,q}^*(x_0)]_\pm = N_\mathbf{p}\ \delta_{m,n}\ \delta_{\mathbf{p},\mathbf{q}} \tag{17}$$

hence, for a *real* κ (positive or negative):

$$[\mathbf{a}_{m,p}(x), \mathbf{a}_{n,q}^*(y)]_\pm = N_\mathbf{p} \exp(ip(y-x)).\exp(i\kappa(c(y,p)-c(x,p)))\ \delta_{m,n}\ \delta_{\mathbf{p},\mathbf{q}} \tag{18}$$

where *x* and *y* are different space-time points. The bracket $[\ ,\ ]_\pm$ refer to Fermi or Bose statistics; $N_\mathbf{p}$ is a normalization factor, usually set equal to *1* (as in (Weinberg 1995)) or to $\sqrt{(\mathbf{p}^2 + M^2)}/M$.
Observe that, if κ is *imaginary*, $\kappa = -i\kappa'$, κ' positive, we get a term like $\exp(-\kappa'(c(x,p) + c(y,p)))$.

3. Quantum Fields

It is possible to define neutral and charged *quantum fields* **φ** and **ψ** (now well defined, operator-valued functions assigned on *X*) as:

$$\boldsymbol{\varphi}(x) = \int(\mathbf{a}_p(x) + \mathbf{a}_p^*(x))\ d\mu(\mathbf{p}) \tag{19}$$

$$\boldsymbol{\psi}(x) = \int \mathbf{a}_p(x)\ d\mu(\mathbf{p}) + \int \mathbf{b}_p^*(x)\ d\nu(\mathbf{p}) \tag{20}$$

$d\mu$ and $d\nu$ are measure on $\mathbf{R}^3$, eventually matrix-valued. Obviously, for κ=0, we shall obtain the usual, singular quantum fields. These are *free fields* (the $P_u^{(INT)}(x)$ term is not considered), but the gravitational effects are fully included in the field operators.
Afterward we can build the Pauli-Jordan functions $\Delta(x, y)$: they depend on $\kappa(c(y,p)-c(x,p))$ and are, generally, not translation invariant. But they are now true, differentiable functions (assigned on X^2) and, for $\kappa=0$, we shall obtain the usual distributions (singular functions), translation invariant. We have, for example, for a real κ:

$$\Delta(x, y) = [\boldsymbol{\varphi}(x), \boldsymbol{\varphi}(y)] = 2 \int \sin(p(y-x)+\kappa(c(y,p)-c(x,p))) \, d\mu(\mathbf{p}) \tag{21}$$

but for an imaginary $\kappa = -i\kappa'$, κ' positive:

$$\Delta(x, y) = [\boldsymbol{\varphi}(x), \boldsymbol{\varphi}(y)] = 2 \int \sin(p(y-x)).\exp(-\kappa'(c(y,p)+c(x,p))) \, d\mu(\mathbf{p}) \tag{22}$$

that leads to an exact, canonical *equal time* commutation relation.

4. Interactions. Self-Energies

4.1. Interaction-Energy Operator

Let us consider a tri-linear *interaction-energy operator* (the Hamiltonian, always in an interaction picture):

$$H^{(INT)}(x) = \lambda \iint(\boldsymbol{\xi}_p^*(x) \, \mathbf{A}_{p,q}(x) \, \boldsymbol{\xi}_q(x)) \, d\mu(\mathbf{p},\mathbf{q}) \tag{23}$$

here $d\mu$ is a measure on $\mathbf{R}^6$; $\mathbf{A}_{p,q}(x)$ is a self-adjoint operator (Bose statistics) related to the gauge fields which mediate the interaction; the operators $\boldsymbol{\xi}_q(x)$ describe the interacting particles (Fermi statistics) and they are linear combinations of the $\mathbf{a}_p(x)$, $\mathbf{a}_p^*(x)$, $\mathbf{b}_p(x)$, $\mathbf{b}_p^*(x)$; λ is a coupling constant. Note that we are only considering the particles interactions and not the interactions between the gauge fields.
When $\kappa=0$ the *interaction-energy operator* (formally) reduces to the standard expression ($x = (t, \mathbf{r})$):

$$H^{(INT)}(t) = \lambda \iint(\boldsymbol{\xi}_p^*(t) \, \mathbf{A}_{p,q}(t) \, \boldsymbol{\xi}_q(t)) \, d\mu(\mathbf{p},\mathbf{q}) \tag{24}$$

time dependent and needing a regularization.

4.2. Particles Self-Masses

If we try to calculate the self-energies (or the *self-masses*) of our particles considering only the one-loop Feynman graphs, we arrive at a **r** (and **g**) dependent expression ((Weisskopf 1939), see also for a more "modern", but essentially equivalent, presentation (Bogoliubov 1980) or (Weinberg 1995)):

$$M^{(SELF)}(\mathbf{r}) = -M.\Lambda.\log(F_{\mathbf{g}}(\mathbf{r})) + const. + \ldots \tag{25}$$

logarithmic diverging when $\kappa=0$ (that is ignoring gravitation). Here M is again the bare mass and Λ a numerical, positive parameter, proportional to the squared coupling constant λ. In the weak gravitational field approximation (subsection 2.3):

$$F_{\mathbf{g}}(\mathbf{r}) = M^{(GRAV)} / r \tag{26}$$

where $M^{(GRAV)}$ is the *effective* (inertial, gravitational) particle mass.

4.3. Counter-Terms

Now we shall assume that the interaction-energy operator contains a mass-like counter-term

$$\int(\boldsymbol{\xi}_p^*(x) \, M^{(CTERM)}(x) \, \boldsymbol{\xi}_p(x)) \, d\mu(\mathbf{p}) \tag{27}$$

and that the real function $M^{(CTERM)}(\mathbf{r})$ (> 0) is proportional to

$$-M.(\Delta k(\mathbf{r}) / k(\mathbf{r})) \tag{28}$$

where M is the bare mass, Δ the Laplace, second order, differential operator and k is a scalar, classical field. Alternatively we can include the counter-term in the free part of the energy operator so the "bare mass" will be a function of x. Presumably the counter-term is of *geometrical origin*, for example it is the Ricci scalar of a (modified) metric tensor **g**.
In any case, imposing the constancy of the sum

$$M^{(SELF)}(\mathbf{r}) + M^{(CTERM)}(\mathbf{r}) \tag{29}$$

(the effective mass cannot depends on **r**), we arrive at a $\mathbf{R}^3$ differential equation (note that here the bare mass do not appears):

$$\Delta k(\mathbf{r}) / k(\mathbf{r}) = -\Lambda'.\log(F_{\mathbf{g}}(\mathbf{r})) + const. \tag{30}$$

For a radial symmetric *k*, rescaling the dependent variable and using the weak field expression of $F_{\mathbf{g}}(\mathbf{r})$, $M^{(GRAV)} / r$,

$$K''(r) + (2/r)\, K'(r) = (\log(r) - w)\, K(r) \tag{31}$$

We are looking for a $K(r)$ going to zero when r diverge, hence we are lead to a countable family of solutions of the above differential equation; the first self-values of w are:

$$1.044;\ 1.847;\ 2.290;\ 2.596;\ 2.830;\ 3.020;\ 3.179 \tag{32}$$

Obviously we can only obtain in this way the *masses ratios* of the first, the second, the third, ... *variety* (or generations) of particles.

5. Leptons Masses

Consider now the case of the electro-weak interactions between *leptons*, mediate by the gauge fields *A, Z, W, W**; M_Z and M_W are the *Z*-particle and *W*-particle masses. To calculate the self-masses we need the three (one loop) Feynman graph (*lept* stands for a *charged* lepton):

$$lept(lept\text{-}photon)lept \ + \ lept(lept\text{-}Zparticle)lept \ + \ lept(neutrino\text{-}Wparticle)lept \tag{33}$$

then in the differential equation of subsection 4.3 appears a sum of three mass-dependent terms:

$$\Lambda_A.\log(M_{lept}) + \Lambda_Z.\log(M_{lept} + M_Z) + \Lambda_W.\log(M_{neutrino} + M_W) \tag{34}$$

The Λ_A, Λ_Z and Λ_W are three positive parameters proportional to the squared electro-weak coupling constants; $\Lambda_A + \Lambda_Z + \Lambda_W = 1$, as a consequence of the rescaling the dependent variable (subsection 4.3). Apparently the experimental data lead to $\Lambda_A/\Lambda_W = 0.231$ and to $\Lambda_A/\Lambda_Z = 0.769$ so:

$$\Lambda_A = 0.151, \quad \Lambda_Z = 0.196, \quad \Lambda_W = 0.653 \tag{35}$$

Finally, assuming that $M_{neutrino} << M_W$, we arrive at:

$$\log(M_\mu / M_{electron}) = 5.32, \quad \log(M_\tau / M_{electron}) = 8.22 \tag{36}$$

in reasonable accord to the experimental values: *5.332* and *8.154* (we have introduced many approximations). But if, for example, we add to the self-masses a $(\log(r))^2$ term (coming from some two loops Feynman graphs), we obtain a quite better accord. It would be also of some interest to consider the not radial symmetric solutions of the *k*-field differential equation.
Obviously we also obtain other, bigger mass values (corresponding, presumably, to highly instable particles):

$$M_4 \approx 11\ Gev, \quad M_5 \approx 34\ Gev, \quad M_6 \approx 72\ Gev, \ \ldots \tag{37}$$

If we try to calculate the neutrinos masses starting from the interaction-energy operator defined at the subsection 4.1, we arrive at a quite problematic result: the neutrinos predicted masses are very large, greater than the *W*-particle and Z-particle masses. To cure this problem it is apparently necessary to modify the structure of the energy operator.

References

Bogoliubov N N, Shirkov D V (1980), Introduction to the Theory of Quantized Fields, John Wiley and Sons
Bourbaki N (2007), Eléments de Mathématique. Variétés différentielles et analytiques: Fascicule de résultats, Springer-Verlag
Lang S (1985), Differential Manifolds, Springer-Verlag
Weinberg S (1972), Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley and Sons
Weinberg S (1995), The Quantum Theory of Fields, Volume I, Cambridge UP
Weisskopf V F (1939), On the Self-Energy and the Electromagnetic Field of the Electron, Physical Review, (56), 72