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UNIMODULAR CONFORMAL AND PROJECTIVE RELATIVITY: AN ILLUSTRATED INTRODUCTION

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Abstract
This is an illustrated presentation of unimodular conformal and projective relativity, a formulation of unimodular relativity in terms of four independent fields with clear physical and geometric interpretations: four-volume element, conformal structure, projective structure, and affine one-form. We present the motivation for the theory, physical and geometrical interpretations of the independent fields, and briefly comment on its applications and prospects for its quantization.

1. Introduction
Unimodular conformal and projective relativity (UCPR) is a formulation of unimodular relativity (UR) in terms of four independent fields with clear physical and geometric interpretation. It is, in a way, an extension of the Palatini approach which takes the metric and the affine connection as fundamental fields. However, these two structures are not irreducible. In UR, the metric can be decomposed into a conformal metric (or *conformal structure*) and a four volume-element, while an affine connection can be decomposed into a projective connection (or *projective structure*) and an affine one-form. UCPR goes a step further by treating the resulting four irreducible fields as independent dynamical fields of the theory.

UCPR was developed as a first step in a search for a background-independent, non-perturbative theory of quantum gravity. We approach this task by first identifying those space-time structures that could be used for measurability analysis of the full inertio-gravitational field. Measurability analysis identifies those concepts that are ideally measurable in the defining context (e.g. concepts of hardness in the context of fluid and solid states of matter in classical thermodynamics) and determines the limits on their measurability and their definability in quantum theory (e.g. position and momentum in QM). The ideal measurement procedure, which may involve any devices and procedures that are consistent with the theory being studied in the regime in question, should yield the limits measurability of the quantities that are in agreement with the uncertanties formally predicted by the theory (Bergmann 1982). Any disagreement between the two indicates an internal inconsistency of the theory. Measurability analysis was successfully conducted by Bohr and Rosenfeld (Bohr 1933), and Bergmann and Smith (Bergmann 1982) in the cases of the quantum electro-dynamics and linearized general relativity (GR), respectively. Since we don't have a quantum theory of gravity, our goal is to use measurability analysis as a guide to a physically motivated formulation of such a theory. The first step in this approach is to formulate a classical theory a gravity that yields itself to measurability analysis. Our work in this direction has led us to UCPR. The choice of conformal and projective structures as fundamental fields is motivated by the work Weyl and Ehlers, Pirani and Schild (E-P-S), to name a few, who emphasized the importance of these structures in GR. In a series of papers, E-P-S showed that conformal and projective structures, and consequently the pseudo-Riemannian space-time geometry in GR, can be axiomatically constructed by considering the propagation of massless and massive particles (Ehlers 1973, Pirani 1973). Being so intimately related to physical fields, conformal and projective structures seem like a good place to start given that we have measurability analysis in mind.

This presentation of UCPR is mostly non-mathematical. It is focused on physical interpretations of the four fundamental fields, as well as various compatibility conditions that can be imposed on them. It is meant to provide physical interpretation and visual description of the structures involved. For a more technical introduction, the reader may look at an earlier publication (Bradonjić 2011). All considerations of this paper deal with a differentiable manifold *M*, a symmetric pseudo-Riemannian metric $g_{\mu\nu}$ of signature (+,-,-,-), and a symmetric affine connection $\Gamma^{\kappa}_{\mu\nu}$. Curvature tensors are formed from their respective connections by the following convention (Schouten 1954): $R_{\dot{\mu}\dot{\nu}\sigma}{}^{\cdot\kappa} = \partial_\mu \Gamma^\kappa_{\nu\sigma} - \partial_\nu \Gamma^\kappa_{\mu\sigma} + \Gamma^\kappa_{\mu\rho} \Gamma^\rho_{\nu\sigma} - \Gamma^\kappa_{\nu\rho} \Gamma^\rho_{\mu\sigma}$.

2. Unimodular conformal and projective relativity

While GR is invariant under the full diffeomorphism group, or the general linear group GL(4, ℝ), UR is invariant under the unimodular diffeomorphisms, or the special linear group SL(4,ℝ), which consists of those point transformations that have a unit determinant. The framework of UR is instrumental in our approach for several reasons which are discussed below. UCPR assumes four independent fields, illustrated in Fig.1: four volume-element, conformal structure, projective structure, and affine one-form.

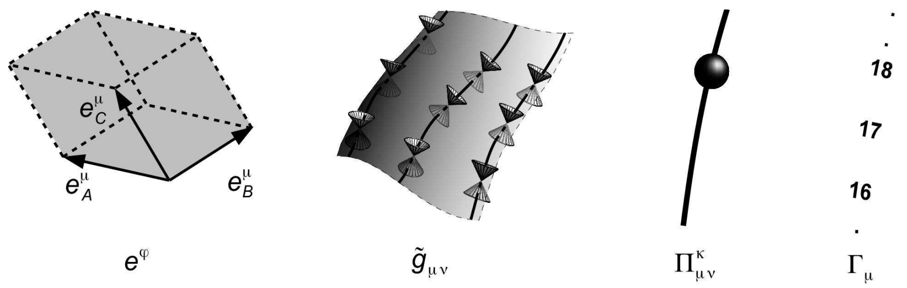

Figure 1: UCPR deals with four independent fields: four volume–element (fourth dimension suppressed), conformal structure, projective structure, and affine one-form.

Four volume-element e^φ is a scalar quantity that is independent of any other space-time structure. As shown in Fig.1(a), it can be interpreted as a volume of a parallelepiped formed by a tetrad of basis vectors in the tangent space at each point of the manifold. Under SL(4, ℝ), the only allowed changes of basis vectors are those that construct a parallelepiped of the same four-volume, and so e^φ is an invariant of the theory. This doesn't mean that e^φ is a fixed quantity, as it is sometimes assumed in other UR theories (Anderson 1975), but that it is same in all allowed frames of reference and independent of any other space-time structures.

Having an invariant e^φ is advantageous for a couple of reasons. First, a four volume-element is essential for measurability analysis because, as Bohr and Rosenfeld showed, only field averages over space-time regions are measurable. The invariance of e^φ guarantees that we can perform space-time integration of all other fields. Second, some approaches to quantum gravity, such as causal set theory (Sorkin 1997), postulate a discrete structure of space-time at high energies and attempt to recover the classical manifold in the low energy limit. In UCPR, quantization of the four-volume may arise from some dynamical procedure, rather than be simply postulated.

Conformal structure, represented by a conformal metric tensor $\tilde{g}_{\mu\nu}$, determines a null cone at each point of M. It distinguishes among spacelike, timelike, and null directions, and determines the causal structure of space-time. Figure 1(b) shows that these null cones globally determine characteristic null wave fronts (three-dimensional null hypersurfaces of constant phase) for any zero-rest-mass radiation field, including electromagnetic and gravitational.

In the same way a metric can be used to construct Christoffel symbols $\{^{\kappa}_{\mu\nu}\}$ and the corresponding metric covariant derivative, $\tilde{g}_{\mu\nu}$ can be used to construct conformal Christoffel symbols $\widetilde{\{^{\kappa}_{\mu\nu}\}}$ and a conformal covariant derivative. Furthermore, we can use $\widetilde{\{^{\kappa}_{\mu\nu}\}}$ to construct the conformal-connection curvature tensor $\tilde{C}_{\dot{\mu}\dot{\nu}\sigma}{}^{\cdot\kappa}$. This curvature tensor is distinct from the Weyl curvature tensor $C_{\dot{\mu}\dot{\nu}\sigma}{}^{\cdot\kappa}$ which is the conformally invariant part of the metric curvature $K_{\dot{\mu}\dot{\nu}\sigma}{}^{\cdot\kappa}$.

A four volume-element and a conformal metric can be combined to form a metric tensor (Schouten 1954),

$$g_{\mu\nu} = e^{\varphi}\tilde{g}_{\mu\nu}. \qquad (1)$$

Under conformal transformations, $\tilde{g}_{\mu\nu}$ remains invariant, while e^{φ} gets mulitiplied by the conformal factor.

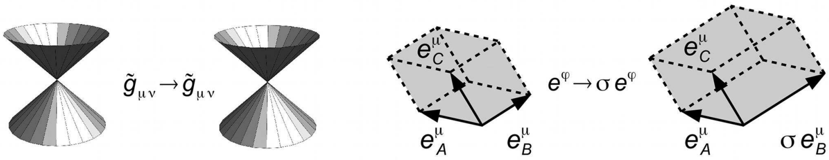

Figure 2: Conformal transformations leave $\tilde{g}_{\mu\nu}$ unchanged and rescale e^{φ}.

Consequently, we recover the expected conformal transformation of the metric. These transformation properties allow us to treat the conformal structure as an equivalence class of conformally related metrics

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Projective structure determines the auto-parallel paths of freely falling massive particles (Fig. 1(c)). Here we note the important distinction between a path and a curve. One can think of a path as an *unparameterized* trajectory through space-time. A curve, on the other hand, is a *parameterized* trajectory. Mathematically, projective structure is represented by projective parameters $\Pi^{\kappa}_{\mu\nu}$. Under GL(4, $\mathbb{R}$), these parameters have complicated transformations laws. T. Y. Thomas first noted that under SL(4, $\mathbb{R}$) $\Pi^{\kappa}_{\mu\nu}$ transform as components of a symmetric, traceless connection (Thomas 1925). This allows us to define projective covariant derivative and the projective-connection curvature tensor $\Pi_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$. $\Pi_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$ is distinct from the usual projective curvature tensor $P_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$ which is the projectively invariant part of the affine curvature tensor $R_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$.

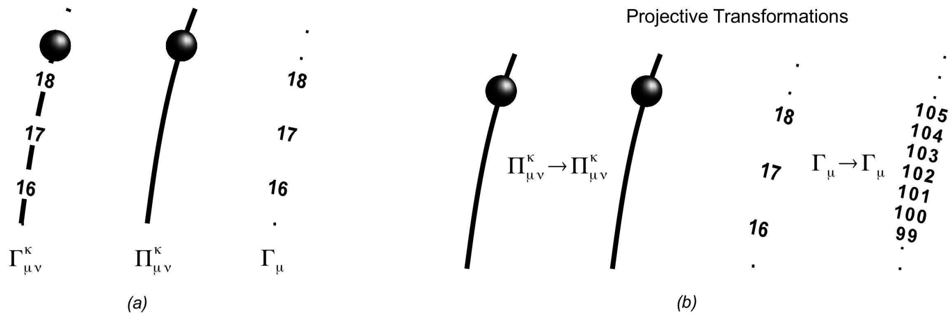

Figure 3: (a) $\Gamma^{\kappa}_{\mu\nu}$ determines curves of freely falling massive particles, $\Pi^{\kappa}_{\mu\nu}$ their unparameterized paths, and Γ_{μ} the parameterization along the paths; (b) Projective transformations leave $\Pi^{\kappa}_{\mu\nu}$ invariant, but change Γ_{μ}.

Affine one-form Γ_{μ} determines a parameterization along a path. As shown in Fig. 3(a), if we strip a curve of its parameterization, we are left with a path. On the other hand, a choice of $\Pi^{\kappa}_{\mu\nu}$ and Γ_{μ} determines a unique affine connection $\Gamma^{\kappa}_{\mu\nu}$ (Schouten 1954),

$$\Gamma^{\kappa}_{\mu\nu} = \Pi^{\kappa}_{\mu\nu} + 1/5\left(\delta^{\kappa}_{\mu}\Gamma_{\nu} + \delta^{\kappa}_{\nu}\Gamma_{\mu}\right). \qquad (2)$$

As shown in Fig.3(b), $\Pi^{\kappa}_{\mu\nu}$ is invariant under projective transformations, while Γ_{μ} is not. The projective connection can be thought of as an equivalence class of projectively related affine connections.

3. Compatibility conditions

In GR, compatibility between $g_{\mu\nu}$ and $\Gamma^{\kappa}_{\mu\nu}$ is imposed by a single condition: $\nabla_{\kappa} g_{\mu\nu} = 0$. UCPR deals with four independent space-time structures, so we can approach the full metric-affine compatibility in steps, as well as impose some new compatibility conditions.

Equi-affine condition demands that e^{φ} be invariant under the parallel transport by $\Gamma^{\kappa}_{\mu\nu}$ (Fig.5 (a)).

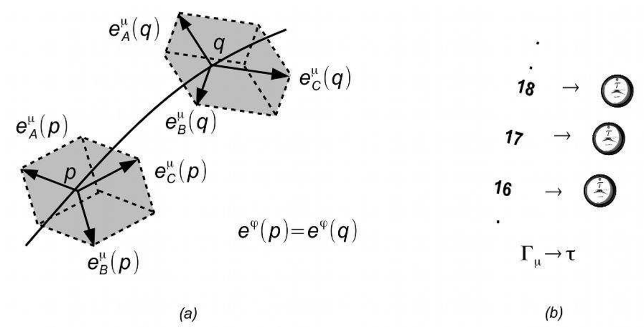

Figure 5: Equi-affine condition (a) demands that the parallel transport by $\Gamma^{\kappa}_{\mu\nu}$ preserves e^{φ}, and (b) allows us to identify the parameterization determined by Γ_{μ} to metrical proper time τ.

In GR, this condition demands that the affine covariant derivative of the metric determinant vanishes, and hence imposes a relation between e^{φ} and $\Gamma^{\kappa}_{\mu\nu}$, and consequently $\Pi^{\kappa}_{\mu\nu}$ and Γ_{μ}. In UCPR we can impose the equi-affine condition without imposing any restrictions on $\Pi^{\kappa}_{\mu\nu}$ by simply demanding that

$$\Gamma_{\mu} = 2\partial_{\mu}\varphi. \tag{3}$$

This condition also allows us to identify the parameterization determined by Γ_{μ} to metrical proper time τ (Fig. 5(b)).

Weyl condition demands that the affine covariant derivative of $g_{\mu\nu}$ be proportional to $g_{\mu\nu}$. As such, it imposes a condition on all four fields of UCPR and, by default, relates the parameterization of auto-parallel curves to proper time τ. Physically, this leads to the second clock effect: the proper time and the ticking rate of a clock depends on its history (Fig.6).

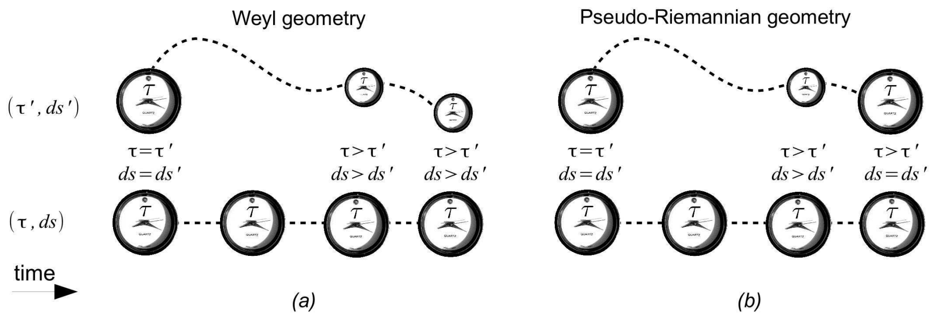

Figure 6: Weyl condition in GR leads to the second clock effect in which the ticking rate of the clock ds depends on its history.

In UCPR, this unnecessary result is avoided by expressing the Weyl condition in terms of $\Pi^{\kappa}_{\mu\nu}$ and $\widetilde{\{^{\kappa}_{\mu\nu}\}}$, without imposing any relation between Γ_{μ} and τ,

$$\Pi^{\kappa}_{\mu\nu} - \widetilde{\{^{\kappa}_{\mu\nu}\}} = \frac{1}{4}\left[\frac{1}{5}\left(\delta^{\kappa}_{\mu}\Gamma_{\nu} + \delta^{\kappa}_{\nu}\Gamma_{\mu}\right) - \tilde{g}_{\mu\nu}\tilde{g}^{\kappa\sigma}\Gamma_{\sigma}\right]. \tag{4}$$

Full conformal-projective compatibility demands that the conformal geodesics are also projective auto-parallel paths, and is satisfied if $\widetilde{\{^{\kappa}_{\mu\nu}\}} = \Pi^{\kappa}_{\mu\nu}$. This condition also puts no restriction on e^{φ} and Γ_{μ}.

Full metric-affine compatibility in UCPR holds if *both* equi-affine and Weyl condition hold. However, having four independent fields allows us to choose which conditions to impose, and naturally allows us to consider theories with intermediate conditions.

4. Conclusion

UCPR provides the flexibility to construct a variety of theories which dynamize one or more of the four independent fields. Lagrangian formulation of UCPR with the standard GR action, but now four independent fields, yields field equations equivalent to those of GR. However, in UCPR we can construct many different Lagrangians that may provide better descriptions of nature.

The next step in the investigation is to determine which space-time structures of UCPR are good candidates for quantization. While there is much to be done in this direction, there are several approaches that seem to be most promising. The obvious candidates for measurability are the various curvature tensors that can be constructed in UCPR. While the conformally invariant curvature tensors $\tilde{C}_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$ and $C_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$ can be probed with zero rest-mass fields. An analogous analysis of the projectively invariant curvature tensors $\Pi_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$ and $P_{\dot{\mu}\dot{\nu}\sigma}{}^{\kappa}$ could be done by using massive particles. Furthermore measurability analysis of these fields can be discussed in terms of fields still attached to its sources, and free fields, both in the near and far zones (Stachel 2012). UCPR also opens numerous possibilities for further investigation. It allows for a construction of a range of theories differing in their choice to dynamize some rather than all four fields, and to impose compatibility conditions that are less restrictive than the full metric-affine compatibility.

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