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String theory and regularisation of space-time singularities.

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Abstract
After a discussion of the general status of space-time singularities in string theory we will concentrate on the case of a singular space-time as seen by an observer moving at the speed of light and we will show that the resulting metric has a very simple universal structure provided that the original space-time had a stress-energy source that satisfies the dominant energy condition. String theory in such singular homogeneous plane wave backgrounds is exactly solvable and the essential details can be described by matrix quantum mechanics with a time-dependent potential. We discuss in some detail the regularisation of this matrix quantum mechanics.

1. Introduction
String theory as a quantum theory of gravity has had a considerable amount of success. It is to date the only known consistent quantum theory of gravity that also includes a more or less standard particle physics picture. In the field of quantum gravity it has also had considerable success in elucidating the interesting phenomenon of black hole thermodynamics. There are precise quantitative results identifying the stringy/gravitational degrees of freedom that give rise to black hole entropy. The study in non-perturbative string theory of scattering from black holes has also led to the extremely powerful conjecture of AdS/CFT holography.

Singularities in General Relativity and the stringy modification thereof, are intuitively signalled by a divergent curvature as one approaches a point, or surface in a space-time. When we say divergent curvature we really mean that some curvature invariant is divergent. However singularities may also arise in other ways, often as an “incompleteness” or conical singularity of the space-time manifold. The “invariant” way to identify many singularities of a space-time manifold is to show that there exists a geodesic which runs to a boundary of the space-time manifold at finite affine parameter.

Singularities have been shown to arise generically for solutions to Einsteins equations - this is the substance of the Hawking-Penrose singularity theorems. The most well studied examples arise in black hole and cosmological space-times and can be broadly classified by their orientation and strength.

- Space-like singularities, the most well-known examples being the big bang of Friedmann Robertson Walker cosmology and that of the Schwarzschild Black Hole.
- Time-like, the classic examples are those inside the inner horizon of Reissner-N\"ordstrom and Kerr(-Newman) black holes. In string theory time-like singularities also arise in the compactified part of the space-time: orbifolds and conifolds are the simplest examples.
- Null - conjectured to arise under generic perturbations of inner horizons. In the context of string theory there has been much recent investigation into the properties of Singular (Homogeneous) Plane Waves which possess a null singularity of particular simplicity.

There are various approaches to studying the physics of singularities.

- Singularities may be resolved in a geometric sense, for instance replacing the region close to and including the singularity by a smooth geometry. Conifold singularities are resolved in this way.
- The metric remains singular, however it may turn out that extra degrees of freedom concentrated at the location of the singularity mean that physical processes in the presence of the singularity remain well-defined.
- There is some other quantum gravity related resolution: Fuzzballs that hide horizons and consequently also singularities; Loop Quantum Gravity has a minimum distance element and thus there is encoded in this theory an upper bound on the space-time curvature.

- Gravitation and space-time enter a non-geometrical phase: a gas of black holes; a new phase of quantum geometry; or an alternative Yang-Mills description of the physics.

There is much that one could say about these different approaches, of which one could also make a complete catalogue. We will concentrate on null singularities and their possible non-geometric resolution via non-perturbative physics in String Theory. For more details refer to (Blau 2011).

2. String theory and singularities

As opposed to the essentially point particles of quantum field theory strings are extended fundamental objects and consequently they may actually “see” singularities differently. It is in fact well known that a variety of time-like singularities can actually be resolved when they are probed by strings rather than point particles. For example, in the case of orbifold singularities (essentially conical singularities) the string itself couples not just to the metric but also to an additional anti-symmetric tensor field. There are additional stringy states that are always attached to the singularities the net effect of which is the absence of a true singularity thus giving rise to the smooth propogation of a string across an orbifold.

In the case of space-like singularities the situation is considerably more complex as one is forced to deal with time-dependent backgrounds and out of equilibrium systems. The time-like singularity of a conifold for example is resolved due to the condensation of non-perturbative states that become massless at the geometrical singularity, in a time-dependent and non-adiabatic system we have very few tools that enable us to consider such phase transitions. In the following we will illustrate how one may neverthless use string theory to study null singularities (as a limit of space-like ones).

Consider a time-dependent metric of the form

$$ds^2 = -f(t)\,dt^2 + g(t)\,dr^2 + r^2\,d\,\Omega_d^2$$

and assume that there is a singularity at *t=0*. All common space-like singularities have such a form where *f(t)* and *g(t)* are to leading order simply powers of *t*. Now to simplify our problem (without hopefully removing all the possibly interesting physics related to singualrities) we will take a particular limit, the Penrose limit, of this space-time.

The Penrose Limit corresponds to zooming into the space-time in a tubular neighbourhood of a null geodesic. The profile of the plane-wave that results upon taking the Penrose limit of a metric with respect to the null geodesic $\gamma(u)$ is

$$ds^2 = -2\,du\,dv + A_{ab}(u)\,z^a z^b\,du^2 + d\,\vec{z}^2$$

with

$$A_{ab}(u) = -R_{aubu}(\gamma(u))$$

where on the right hand side we have frame components of the curvature tensor of the original metric evaluated along the null geodesic.

Thus, the Penrose limit is actually encoding some physical information about the original metric giving an exact description of the space-time along the null geodesic. The geometrical significance of the wave-profile $A_{ab}(u)$ is that it is the transverse null geodesic deviation matrix along $\gamma(u)$ of the original metric

$$\frac{d^2}{du^2}Z^a = A_{ab}(u)Z^b$$

where *Z* is the transverse geodesic deviation vector. More precisely the Penrose limit encodes the information about tidal forces along the corresponding null geodesic in the original geometry. For a singular homogeneous plane wave we have

$$A_{ab}(u) = \frac{1}{u^2}A_{ab}$$

and the SHPW's have a divergent tidal force thus retaining this important feature of the singular gravitational field in the original geometry as *u→0*.

In (Blau 2004) we demonstrated that:
Penrose Limits of spherically symmetric space-like or time-like singularities of power-law type satisfying (but not saturating) the Dominant Energy Condition (DEC) are singular homogeneous plane waves with profile

$$A_{ab}(u) = -\omega_a^2 \delta_{ab} u^{-2}$$

3. Yang-Mills from Discrete Light Cone Quantization

The Discrete Light Cone Quantization construction applied to string theory (Sen , Seiberg) is naturally adapted to SHPW's due to their symmetries and geometrical structure. Applied to the metric

$$ds^2 = -2\,du dv + A_{ab} z^a z^b \frac{du^2}{u^2} + d\vec{z}^2$$

and expanding the resulting Dirac-Born-Infeld D-string action around a classical solution, one finds that the fluctuations around this trajectory are described by the action

$$S = \frac{1}{2}\int d^2\sigma\left(-\eta^{\alpha\beta}\partial_\alpha z^a \partial_\beta z_b + A_{ab}(t) z^a z^b + \frac{1}{2} g_{YM}^2 \left[z^a, z^b\right]^2\right)$$

and the Yang-Mills coupling is related to the original dilaton

$$g_{YM} \sim \frac{1}{g_s l_s} e^{-\varphi}$$

In the following we concentrate on negative frequency squared for which case we have a strong string coupling singularity corresponding to weak Yang-Mills coupling. Thus there is hope that strongly coupled string theory may have an alternative description in terms of a weakly coupled Yang-Mills theory. An analysis of classical and quantum mechanics with this lagrangian (O'Loughlin 2010) shows that near the singularity the typical quartic interaction is not important compared to the time dependent mass-term.

An attractive picture suggested in the original paper by CSV, was that near the singularity the weak coupling should give rise to a highly non-geometric picture of the space-time, where the D-string position is no longer described by a set of commuting coordinates but rather by a set of non-commuting matrices. Furthermore there was a significant hope that this new picture would lead to a tractable picture of the near singularity physics in terms of these non-singular and non-geometric fields. However suggestive this picture, it has been notoriously difficult to make this quantitative.
Nevertheless the general analysis of string theory in the singular plane-wave background, following classical and semi-classical reasoning (O'Loughlin 2010) for dynamics in the time-dependent potential leads one to consider the considerably simpler lagrangian (*t=u*)

$$S_{bc}=-\frac{1}{2}\int d\sigma^2\left(\eta^{\alpha\beta}\partial_\alpha z^a\partial_\beta z^b-\omega_a^2(t)z^a z^a\right)$$

To analyse this we will restrict to *1+1* spacetime dimensions and decompose into Fourier modes in the spatial σ worldsheet direction, leading to an infinite number of quantum mechanical systems labelled by the level *n*

$$S_{bc}=-\frac{1}{2}\int dt\left(\dot{z}^2-\omega_n(t)^2 z^2\right)$$

where

$$\omega_n(t)^2=\frac{a(1-a)}{t^2}+kn^2$$

and a appears also in the corresponding dilaton field

$$\varphi(t)=-4\mathrm{a}(a-1)\log(|t|)$$

One can easily show by basic quantum mechanics that this system is singular at $t\to 0$. Is there an (essentially) unique way to propagate this system through the singularity?

4. Regularisation of the singularity

A geometrical regularisation of the singularity consists in replacing the singular frequency by a smooth function with a parameter ε such that when the parameter is taken to zero this function returns to the original singular one. The simplest requirement that such a regularisation should satisfy is to provide a finite propagator between a time preceeding the singularity to a time in the future of the singularity. It turns out that a good regularisation (not the only possibility) is

$$\omega_\epsilon(t)^2=a(1-a)\frac{\left(t^2-\beta\epsilon^2\right)}{\left(t^2+\epsilon^2\right)^2}$$

and the corresponding propagator is

$$\lim_{\epsilon\to 0}F(t;s)=\frac{1}{1-2\mathrm{a}}\left(q\left(t^a\right)\left(s^{1-a}\right)+q^{-1}\left(s^a\right)\left(t^{1-a}\right)\right)$$

where $q=\pm 1$. This result is interesting in that it does not correspond to a naïve analytic continuation of the original propagator, however it is also unsatisfactory as it does not provide any further intuition about the possible physical mechanism that allows propagation through the origin. To further investigate this question we also looked at the propagator between any finite *t* and the origin finding that it is always singular. More elaborate regularisations can also make this propagator *F(t;0)* finite but at the expense of introducing additional free parameters into the final result. One of the main hopes of this study was that one could regularise in such a way that there are no (or at most just one) free parameter so we deem this situation to be unsatisfactory. We can still read off some general lessons about the behaviour of strings near null singularities and we will discuss these in the following section.

5. Discussion

For the case of most interest, with strong string coupling divergent at *t=0*, we find that there is a good regularisation but it is still not completely clear how this regularises the physics as the string grazes the singularity and runs off to infinity and thus out of the Penrose Limit of the original spacetime. This seems to indicate that either the Penrose limit, or the DLCQ procedure is removing some degrees of freedom that may be important at large *z,* causing the resulting physics to remain singular. In addition, in taking these limits one needs to confront various problems with the order of limits and these need to be analysed more carefully.

The Penrose limit is indeed the leading term in an expansion of the space-time metric around a null geodesic, the Penrose-Fermi expansion, and it would be a useful exercise to study the next terms in this expansion to investigate further the possibility that the singularity may be removed be some additional degrees of freedom.

As mentioned above the original conjecture of CSV (Craps 2005) involved a highly non-Abelian phase of the 1+1 dimensional Yang-Mills theory, a phase that is very weakly coupled. At this point one should also attempt to construct alternative models that enable an analytic treatment of this highly non-geometric phase given that our current and simple minded approach does not lead to a resolution.

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