Dark Energy from Curvature and Ordinary Matter Fitting Ehlers-Pirani-Schild: Foundational Hypotesis

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We discuss in a critical way the physical foundations of geometric structure of relativistic theories of gravity by the so-called Ehlers-Pirani-Schild formalism. This approach provides a natural interpretation of the observables showing how relate them to General Relativity and to a large class of Extended Theories of Gravity. In particular we show that, in such a formalism, geodesic and causal structures of space-time can be safely disentangled allowing a correct analysis in view of observations and experiment. As specific case, we take into account the case of $f(R)$ gravity.

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I. INTRODUCTION

Einstein General Relativity (GR) is a self-consistent theory that dynamically describes Space, Time and Matter under the same standard. The result is a deep and beautiful scheme that, starting from some first principles, is capable of explaining a huge number of gravitational phenomena, ranging from laboratory up to cosmological scales. Its predictions are well tested at Solar System scales and give rise to a comprehensive cosmological model that agrees with the Standard Model of particles, with the recession of galaxies, with the cosmic nucleosynthesis and so on. Despite these good results, the recent advent of the so-called Precision Cosmology tests from astrophysics (rotation curves of galaxies) and possible some tests coming from the Solar System outskirts (e.g. the Pioneer anomaly) entail that the self-consistent scheme of GR seems to disagree with an increasingly high number of observational data, as e.g. those coming from IA-type Supernovae, used as standard candles, large scale structure ranging from galaxies up to superclusters. Furthermore, being not renormalizable, GR seems to fail to be quantized in any classical way (see [4]). In other words, it seems, from ultraviolet up to infrared scales, that GR is not and cannot be the definitive theory of Gravitation heven if it successfully addresses a wide range of phenomena.

Many attempts have been therefore made both to recover the validity of GR at all scales, on one hand, and to produce theories that suitably generalize Einsteins one, on the other hand. In order to interpret a large number of recent observational data inside the paradigm of GR, the introduction of DarkMatter (DM) and Dark Energy (DE) seemed to be necessary: the price of preserving the *simplicity* of the Hilbert Lagrangian has been, however, the introduction of rather odd-behaving physical entities which, up to now, have not been revealed by any experiment at fundamental scales. In other words, we are observing the large scale effects of missing matter (DM) and the accelerating behaviour of the Hubble flow (DE) but no final evidence of these ingredients exists, if we want to deal with them as standard quantum particles or fields. However, from an observational point of view, considering GR + cosmological constant+DM gives an extremely good snapshot of the currently observed Universe. The problem is that dynamics of previous epochs cannot be reconstructed and addressed in a self-consistent way starting from the present status of observations. Furthermore, it seems that the type of DM to be considered strictly depends on the size of selfgravitating structures (e.g. the dynamical behavior of DM in small galaxies, in giant galaxies and in galaxy clusters is completely different). So, besides the issue to find out DM and DE at fundamental scales, it seems hard to find out a general dynamics involving such components working at all cosmic epochs and at any astrophysical size. With these considerations in mind, one can wonder if extending gravity sector could be a more economic and useful approach which does not involve too much exotic ingredients but retains all the good results achieved by GR (for a review, see *e.g.* [5]). In this paper we address some of the recent issues concerning the geometrical structure of "physically reasonable" gravitational theories, starting from the fundamental work of Elehers-Pirani-Shild [6–8] about the geometric and physical foundations of relativistic theories of gravitation and revisiting them, *á la Palatini*, in view of applications to the new challenges discussed above [9, 10].The outline of the paper is as follows. In Section II we introduce the EPS framework. Section III is devoted to the EPS formalism in GR while Section IV is a critical discussion of such an approach. In Section V, we discuss EPS from the point of view of Extended Theories of Gravity (ETG). In particular, the straightforward extension of GR, $f(R)$-gravity, is taken into account. Section VI and VII is devoted to discussion and conclusions. A new paradigm for gravitational theories is proposed assuming the EPS paradigm.

II. EHLERS-PIRANI-SCHILD-THEORY

We first summarize Ehlers-Pirani-Schild (EPS) analysis of the mathematical structures that lie at the basis of all "reasonable" relativistic gravitational theories [6–8]. In early 70s EPS started from a set of well motivated physical property of light rays matter in a relativistic framework to derive the geometrical structure of space-time from potentially observable objects. This is particularly suitable to discuss which geometric structure is observable and which are conventionals. In this way it provides stronger physical motivation and understanding not only of space-time geometry as such, but also in comparison with more general geometries (as candidates for mathematically modeling physical space-time). EPS specifically highlighted the potential role of space-time models based on Weyl geometry. Supplying this new axiomatic characterization of the otherwise mathematically familiar space-time geometry structure, EPS also brings relevant new insight even from a strictly mathematical (geometrical) standpoint. Einstein's GR uses ses advanced mathematical ideas. Things like 4-dimensional curved spacetime are not easy to grasp. Even if one masters the math behind it, the essential physical meaning and content is not obvious. EPS is one of a series of attempts to clarify the physics behind the math. Unfortunately and unavoidably, getting there requires even more abstract math. At first sight, this seems self-defeating; however, some of these mathematical ideas are chosen so as to be closer to 'operational' physical interpretation, representing more elementary physical observation, measurement and construction. In the upshot, EPS ends up with Lorentzian metric ($L4$), rather then of accepting it as starting point: the idea is to rebuild $L4$ from scratch, using only bricks with intuitively clear physical meaning to the extent possible, and at the cost of some extra math. For example as far as metric structure $L4$ is concerned EPS clearly showed that what is physically well defined is a conformal structure (the class of all a conformally equivalent Lorentzian metrics) such a representative g (*i.e.* a specific Lorentzian metric) can be singled out only by convention of an observer. As is typical in axiomatic reconstructions like EPS, one exploits the benefit of hindsight, as the intended result (in this case: $L4$ spacetime of General Relativity) is already known. So this in no way detracts from Einstein's original feat, on the contrary. The scope of EPS is limited to the kinematics of space-time itself; the problem of any possible axiomatic derivation or reconstruction of Einstein field equations (thet is dynamics) governing matter and gravity within such a space-time model, is left open.

"The approach shows how quantitative measures of time, angle and distance, and a procedure of parallel displacement... can be obtained constructively from 'geomtry free' assumptions about light-rays and freely falling particles; pseudo-Riemannian (or Weylian) geometry is recognized even more clearly than before as the appropriate language for a generalized kinematics which allows for the unavoidable and ever-present 'distortions" called gravitational fields." (Ehlers)

A. Outline of Ehelers-Pirani-Shild construction

With the above considerations in mind, let us outline the main points of EPS conceptual construction. The construction of EPS space-time proceeds in steps as sketched below, each one enriching the axiomatic content of the underlying set of events. Roughly, the underlying idea is the following. From differential geometry, one knows that the geodesics determine 'their affine connection' (assuming torsion to be zero, for instance) and hence a corresponding metric. Now, in contrast to the metric itself, these geodesics do possess an immediate physical interpretation (as light ray worldlines for null geodesics or particle world lines for timelike ones). So in very general terms, one tries to reconstruct the sought-after metric from known geodesics that fulfill certain qualitative criteria (postulates), which are themselves physically meaningful and plausible.

- **Particles and light rays in event space**.

  EPS adopts a set $\mathcal{M}$ of events (to become the space-time manifold) as its backdrop. On this, a set of particles p and a set of light rays l are assumed given. Each particle and each light ray are identified with their 'world line' of events.

- **Smooth radar coordinates for events**

  As subsets of the space of events, particle and light ray world lines are taken to be smooth one dimensional manifolds. A permissible local coordinate represents time as measured by a (possibly irregular) local clock. Light ray messages between particles p and q smoothly relate their private time parameters, the timing of echoes received back by p also relate smoothly to that of the message flashes it sent out to q to begin with. Using 'radar soundings' in this way, pairs of 'observer' particles set out to map surrounding events by assigning 2 time values each, or a total of 4 coordinates each. Postulating that this process may cover the entire event set, the events form a smooth 4-dimensional space (manifold).

- **Light propagation ensures local validity of pointwise causality**

  At each point of space-time (event), the propagation of light determines an infinitesimal null cone, amounting to a conformal structure $\mathcal{C}$ of Lorentzian signature. This assertion is stated operationally, demanding that one may (topologically) distinguish between $\mathcal{C}$-time-like, space-like and null vectors, directions and curves at an event. Null curves lying on a null hypersurface are singled out as null geodesics.

- **Free falling particles encode influence of gravity on particle motion** Among the timelike curves, the free-falling particles form a preferred family of wordlines. Imposing a generalized law of inertia provides a projective structure, with free-fall world lines as its ($\mathcal{C}$-time-like) geodesics.

- **Fre fall implicitly define a projective structure $\mathcal{P}$.**

  They in turns determine, by a canonical gauge fixing, a preferred connection space-time.

- **Light and particle motion agree** Then one can define two compatible conformal and projective structures on space-time. The choice of representatives is a conventional gauge fixings. The conventional nature of metrics and connections is important to be noticed in view of which quantities are to be considered physically sound. In particular, one can choose canonically a standard representative of projective structure imposing

$$\nabla^{(\Gamma)}_{\mu} g_{\alpha\beta} = 2V_{\mu} g_{\alpha\beta} \tag{1}$$

  for some covector V. Then there is a canonical connection $\Gamma^{\alpha}_{\beta\mu}$ which, of course, depends on extra degrees of freedom depending on A. The triple $(\mathcal{M}, \mathcal{C}, \Gamma)$ is called a *Weyl-geometry*. It is called *metric* if there exists a representative $g \in \mathcal{C}$ such that $\Gamma = \{g\}$ coincides with the Christoffel symbols of the metric g. In this case, the metric g describes light rays and particles free fall, as it is assumed in standard GR. However, in general one needs two different (still compatible) structures to describe light rays and matter free fall. Let us stress once again that there is no reason at this stage to assume that the Weyl-geometry obtained on space-time is metric. A Weyl space possesses a unique affine structure $\mathcal{A}$: $\mathcal{A}$ geodesics are $\mathcal{P}$ and $\mathcal{A}$ parallel displacement preserves $\mathcal{C}$ nullity. In a Weyl space, one may construct a “proper time” arc length (up to linear transformation) along non-null curves by purely geometrical means (*i.e.* using light rays reflected from particles only, so without any need for atomic clocks). In technical terms, one employs affine parallel displacement, and congruence in the tangent space, as defined by $\mathcal{C}$. This 'geodesic' clock is known as the Marke-Wheeler clocks [11].

B. Hypotesis

In summary EPS analysis is based on a number of assumptions:

It physically distinguishes the Principle of Equivalence from the Principle of Causality and investigates the need of measuring and describing SpaceTime structure through light rays. The need of measuring in SpaceTime and using light rays requires that SpaceTime carries a (Lorentzian) metric while the Principle of Equivalence and interaction with matter ("Free Fall" under gravitational pull) requires that SpaceTime carries also a (Linear or Affine) Connection. The Connection, an object that can be reduced to be zero at each single point, is the potential of the gravitational field. The Metric determines causality and photon propagation. According to EPS analysis, in order for a Gravitational Theory being physically reasonable, compatibility conditions should exist between the Metric and the Connection. The Connection defines a family of autoparallel lines (also called improperly geodesics). They establish the free fall of point-like (in principle massive) "test particles". The Metric defines light cones and a family of geodesics. Null geodesics of the Metric are paths of light rays (photons). The family of autoparallel lines of the Connection determine an equivalence class of "Projectively Equivalent Connections". Along them free fall is the same, only proper time changes. The light cones of the Metric define an equivalence class of Conformally Equivalent Metrics. Along them units and measuring devices change point by point, but light rays and photon trajectories are the same. The required compatibility condition amounts to pretend that the two families of autoparallel lines of the (projective equivalence class of) Connections and the family of null geodesics of the conformal equivalence class of Metrics are in a precise relation: each null geodesic of the Metric has to be one of the autoparallel lines of the Connection At this point of their fresh analysis all Foundational Axioms have been satisfied. In order to recover General Relativity as the Unique Relativistic Theory of Gravitation Elhers, Pirani and Schild make some further axiomatic hypotheses:

- **Speed of time does not depend on path**

  A final physical assumption (expressed mathematically as an axiom) ensures the existence of a Lorentzian metric, which determines both light cones and free fall.

"Equally spaced clock ticks" along one particle world line are transported to a nearby particle by Einstein simultaneity. Imposing that this must generate (approximately) equidistant ticks also for the second particle and applying the equation of geodesic deviation for the curvature tensor given by $\mathcal{A}$ implies (through the vanishing of the Weyl 'track curvature') the existence of a single Lorentzian metric compatible to both $\mathcal{C}$ with $\mathcal{A}$.

This finally 'reduces' Weyl space to $L4$. Requiring in this way that 'time runs equally fast along all paths' amounts to denying the existence of a 'second clock effect'. Indeed, in (Lorentzian) GR, only the 'time interval' between 2 events is path dependent (*i.e.* the 'first clock effect'); not the 'speed' of time.

- *"Metricity Axiom"*: a single Metric is chosen in the conformal class and the Connection is chosen while be the Levi-Civita Connection of this Metric.

With this above hypothesis the compatibility conditions are met. Notice, however, that this just amounts to say that the gravitational theory is of "purely metric" nature. To recover GR as the unique Relativistic Theory of Gravitation one has in fact to make a further assumption.

III. ELEHERS-PIRANI-SCHILD AND GENERAL RELATIVITY

In order to recover General Relativity as the Unique Relativistic Theory of Gravitation one has in fact to make the following further axiomatic hypotheses:

"Lagrangian Axiom": the Lagrangian that governs gravitational field equations (in absence of Matter) is the Scalar Curvature.

The *"Metricity Axiom"* has in fact no real physical grounds. According to EPS (and to physical needs) a Metric has to exist to define rods and clocks, but there is no need to pretend from the very beginning that it defines also the gravitational potential, *i.e.* the Connection. Assuming that the Metricity Axiom holds is just a "matter of taste" and in a sense it corresponds to have a great mathematical simplification. From the viewpoint of Lagrangian Mechanics it is a purely kinematical restriction imposed *a priori* on Dynamics. Physically spacing, it is much better not to impose *a priori* purely kinematical restriction on Dynamics. Physics requires that possible restrictions should be obtained from dynamics rather than imposed *a priori* as a constrains. This point was perfectly clear to Albert Einstein when, in 1923, he tried to establish a more general setting for Gravity (and Electromagnetism) by assuming *a priori* that both a Metric and a Connection must be chosen, from the beginning, as dynamical variables. **So-called "Palatini formalism" was born**. Also the "Lagrangian Axiom" had in fact no real physical grounds. Once it is clear which are the variables that have to enter dynamics, the choice of a Lagrangian for them is again a "matter of taste" or it should be at least determined on the basis of Phenomenology, in order to fit observational data. When Hilbert, in 1916, in the purely metric framework (the only one that was available before 1919 and Levi-Civitas work on Linear Connections) assumed the Lagrangian to be the Scalar Curvature of the Metric this was, in a sense, an obliged choice. Dictated by "simplicity". The choice of the Hilbert-Einstein Lagrangian $R(g)$, made in 1916, was not only the "simplest one". It satisfied the will to obtain second-order field equations suitably generalizing Newtons law, fitting all astronomical predictions, satisfying conservation of matter and being compatible with Maxwell.

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = kT_{\alpha\beta}\,, \qquad (2)$$

where $G_{\alpha\beta}$ is the Einstein tensor, a combination of curvature invariants derived from Bianchi's identities, $T_{\alpha\beta}$ is the stress-energy momentum tensor and $\kappa = 16\pi G$ is the gravitational coupling constant.

In the new framework introduced by assuming both metric and connection among the variables, Einstein decided to take into account again, in 1923, the Lagrangian to be the scalar curvature (of metric and connection), again for the sake of simplicity. At that time there were very few observations fine enough to be used as tests and all of them agree with purely metric predictions. Thus the first test for an extended theory was to reproduce standard GR in purely metric formalism. When Einstein, in 1923, in the new framework he introduced by assuming both a Metric and a Connection among the variables he decided to assume again the Lagrangian to be the Scalar Curvature of the Metric and the Connection, again for the sake of simplicity. In this new framework and with the Linear Lagrangian $R(g, \Gamma)$ he proved that no realy new Physics comes on stage. Field equations impose in fact, *a posteriori*, hat the Connection is nothing but the Levi-Civita Connection of the Metric, so that GR is eventually recovered. Einstein did not investigate, however, what happens when Matter is coupled to the Linear Lagrangian $R(g, \Gamma)$. In this case just a few slight changes are necessary if Matter couples with the Metric g but great difficulties arise if Matter couples with the Connection Γ (as it should). Around the sixties a number of mathematical papers were written about possible generalizations of Einsteins Theory by reverting to Non-Linear Lagrangians, more complicated than $R(g)$. These Higher Order Theories remained just as a mathematical game for long time. Renewed interest towards Non-Linear Lagrangians more complicated than R(g) (Higher Order Theories) was lately determined by new phenomenology, such as: Inflation, Acceleration in the Expansion, Dark Matter, Quantum Gravity, Low Energy Limit of String Models [12, 13].

IV. ELEHERS-PIRANI-SCHILD REVISED FORMALISM

The analysis of EPS – concerning the mathematical and physical foundations of relativistic theories of gravitation and the compatibility between (conformal classes of) metrics and (projective classes of) connections – is worth of being revisited. EPS have shown that the family of gravitational theories that satisfy all of their Axioms (with the exception of the "Metricity Axiom" and the "Lagrangian Axiom") includes many (but not all) of the currently investigated frameworks for (relativistic) gravitation. First of all, it suggest that the correct and most general framework for dealing with gravity is the Palatini formalism, since it is based on the physical and mathematical distinction between the Principle of Equivalence and the Principle of Causality that for obvious reasons are mathematically and physically distinct. They imply the necessity of introducing *a priori* distinct and separate structures to full fill them, even if compatibility is required *a posteriori* on the mathematical and physical structures they induces on space-time. Within this formalism, the most general class of theories that care be considered without renauncing to the physical requirements point out by EPS analysis, is the family of so-called "Further Exteneded Theories of Gravity" that has been explicitly introduced in [14]. This class includes all gravitational theories in which the gravitational Lagrangian depend on g and the (Ricci) curvature of the connection, the matter Lagrangian interacts allows in principle interaction of matter with both g and Γ and, *a posteriori* or *a priori*, field equations imply EPS compatibility. Of course one is free to work in more general frameworks for gravitation, but in such a case one has to remind that at list one EPS requirements will fail.

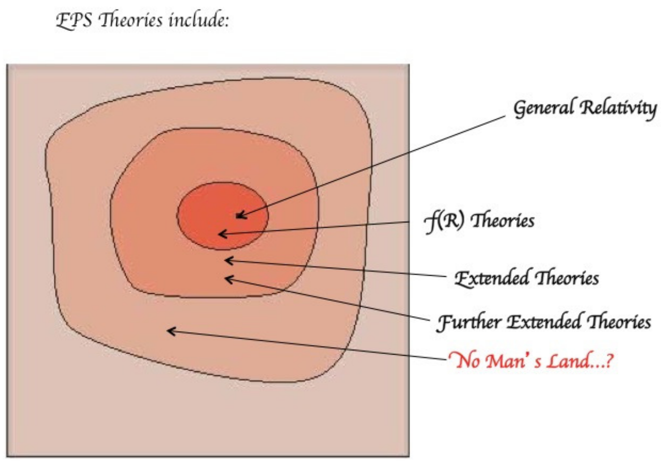

FIG. 1. the eps theories

Our choice wil be more restrictive and will be therefore based on three assumptions:

1. assume Palatini - EPS framework and accept the view that in Palatini formalism the gravitational field is encoded in to the dynamical connection (*i.e.* free fall) while the dynamical metric has more to do with measures, rods, clocks and causality. We accept moreover that dynamics, and in particular the interaction with matter, will determine *a posteriori* the relations between the metric and the connection, so to satisfy EPS compatibility requirements.

2. assume that the Lagrangian is a possibly non-linear function of curvature of g and Γ;

3. assume that the Lagrangian is "simple" - the simplest choice is of course $f(R(g,\Gamma)$ but it is not the only simple case.

V. THE EHELERS-PIRANI-SCHILD APPROACH FOR $f(R)$-GRAVITY

With this choice one can show that if no Matter is present, purely gravitational equations entail that the Connection Γ entering dynamics is still the Levi-Civita Connection of the Metric g while a (quantized) Cosmological Constant enters the game and somehow determines the asymptotic freedom for Gravity. One should remark that if Matter is present and couples only to the Metric g things change if and only if the trace t of the Stress Tensor is different from zero. In particular, thence, nothing changes when Electromagnetism couples, so that the light cone structure and photons are not affected when passing to Palatini framework. It is however known that - both in purely metric formalism (higher order gravity) and in the Palatini approach (first order gravity) - coupling with matter generates relativistic effects that are not present in vacuum. This is particularly evident when one relies on non-linear Lagrangians of the type$f(R)$. In $f(R)$ gravity in the Palatini approach in presence of Matter coupled only with g field equations still imply that the Connection is metric, but now it is the Levi-Civita Connection of a new Metric h, conformally related with the Metric g given in the Lagrangian.

Being this the core point of our discussion, we want to derive in details the field equations of $f(R)$ gravity in Palatini formalism and then perform the EPS analysis in this framework.

Let us first consider on $\mathcal{M}$ a metric field g, a torsionless connection Γ and a generic tensor density A of rank 1 and weight -1. The covariant derivative of A_μ is then defined as

$$\overset{\Gamma}{\nabla}_\mu A_\nu = d_\mu A_\nu - \Gamma^\lambda_{\nu\mu} A_\lambda + \Gamma^\lambda_{\lambda\mu} A_\nu \,, \tag{3}$$

Accordingly, we have

$$\begin{aligned}\overset{\Gamma}{\nabla}_{(\mu} A_{\nu)} &= d_{(\mu} A_{\nu)} - \left(\Gamma^\epsilon_{\nu\mu} - \delta^\epsilon_{(\nu}\Gamma^\lambda_{\mu)\lambda}\right) A_\epsilon = \\ &= d_{(\mu} A_{\nu)} - u^\epsilon_{\mu\nu} A_\epsilon \,, \end{aligned} \tag{4}$$

where we set $u^\epsilon_{\mu\nu} := \Gamma^\epsilon_{\mu\nu} - \delta^\epsilon_{(\mu}\Gamma^\lambda_{\nu)\lambda}$.

Let us consider the following Lagrangian (density)

$$\mathcal{L} = \frac{1}{\kappa}\sqrt{g} f(R) + g g^{\mu\nu} \overset{\Gamma}{\nabla}_\mu A_\nu \,, \tag{5}$$

where $g = |\det(g_{\mu\nu})|$, $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$ is the scalar curvature of (g,Γ), and $f(R)$ is a generic (analytic) function. By variation of this Lagrangian and usual covariant integration by parts one obtains

$$
\begin{aligned}
\delta\mathcal{L} = & \frac{\sqrt{g}}{\kappa}\left(f'(R)R_{(\alpha\beta)} - \frac{1}{2}f(R)g_{\alpha\beta} - \kappa T_{\alpha\beta}\right)\delta g^{\alpha\beta} + \\
& -gg^{\alpha\beta}A_\lambda \delta u^\lambda_{\alpha\beta} + \frac{\sqrt{g}}{\kappa}g^{\alpha\beta}f'(R)\overset{\Gamma}{\nabla}_\lambda\ \delta u^\lambda_{\alpha\beta} + \\
& +gg^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\ \delta A_\nu = \\
& = \frac{\sqrt{g}}{\kappa}\left(f'(R)R_{(\alpha\beta)} - \frac{1}{2}f(R)g_{\alpha\beta} - \kappa T_{\alpha\beta}\right)\delta g^{\alpha\beta} + \\
& -\frac{1}{\kappa}\left(\overset{\Gamma}{\nabla}_\lambda\left(\sqrt{g}g^{\alpha\beta}f'(R)\right) + \kappa g g^{\alpha\beta}A_\lambda\right)\delta u^\lambda_{\alpha\beta} + \\
& -\overset{\Gamma}{\nabla}_\mu\left(gg^{\mu\nu}\right)\delta A_\nu + \\
& +\overset{\Gamma}{\nabla}_\lambda\left(\frac{\sqrt{g}}{\kappa}g^{\alpha\beta}f'(R)\delta u^\lambda_{\alpha\beta} + gg^{\lambda\nu}\delta A_\nu\right), \qquad (6)
\end{aligned}
$$

where we used the well-known identity $\delta R_{(\alpha\beta)} = \overset{\Gamma}{\nabla}_\lambda\ \delta u^\lambda_{\alpha\beta}$ and we set for the energy-momentum tensor $T_{\alpha\beta} := \sqrt{g}\left(g_{\alpha\beta}g^{\mu\nu}\overset{\Gamma}{\nabla}_\mu A_\nu - \overset{\Gamma}{\nabla}_{(\alpha} A_{\beta)}\right)$.

Field equations are

$$
\begin{cases}
f'(R)R_{(\alpha\beta)} - \frac{1}{2}f(R)g_{\alpha\beta} = \kappa T_{\alpha\beta}\,, \\
\overset{\Gamma}{\nabla}_\lambda\left(\sqrt{g}g^{\alpha\beta}f'(R)\right) = \alpha_\lambda\sqrt{g}g^{\alpha\beta}f'(R)\,, \\
\overset{\Gamma}{\nabla}_\mu\left(gg^{\mu\nu}\right) = 0\,,
\end{cases} \qquad (7)
$$

where we set $\alpha_\lambda := -\kappa\frac{\sqrt{g}}{f'(R)}A_\lambda$. Notice that the third equation (that is the matter field equation) is not enough to fix the connection due to the contraction. Notice also that these are more general than field equations of standard $f(R)$ theories due to the rhs of the second equation (that is originated by the coupling between the matter field A and the connection Γ). Nevertheless one can analyze these field equations along the same lines used in $f(R)$ theories. Let us thence define a metric $h_{\mu\nu} = f'(R)g_{\mu\nu}$ and rewrite the second equation as

$$
\overset{\Gamma}{\nabla}_\lambda\left(\sqrt{h}h^{\alpha\beta}\right) = \alpha_\lambda\sqrt{h}h^{\alpha\beta}\,. \qquad (8)
$$

According to the analysis of EPS-compatibility done in [14] this fixes the connection as

$$
\Gamma^\alpha_{\beta\mu} := \{h\}^\alpha_{\beta\mu} - \frac{\kappa}{2f'(R)}\left(h^{\alpha\epsilon}h_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)a_\epsilon\,, \qquad (9)
$$

where for notational convenience we introduced the 1-form $a_\epsilon := \sqrt{g}A_\epsilon$. For later convenience let us notice that we have

$$
K^\alpha_{\beta\mu} \equiv \Gamma^\alpha_{\beta\mu} - \{h\}^\alpha_{\beta\mu} = -\frac{\kappa}{2f'(R)}\left(h^{\alpha\epsilon}h_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)a_\epsilon \qquad (10)
$$

Now we can define the tensor $H^\alpha_{\beta\mu} := \Gamma^\alpha_{\beta\mu} - \{g\}^\alpha_{\beta\mu}$ and obtain

$$
\begin{aligned}
H^\alpha_{\beta\mu} = & K^\alpha_{\beta\mu} - \frac{1}{2}\left[g^{\alpha\lambda}g_{\beta\mu} - 2\delta^\lambda_{(\beta}\delta^\alpha_{\mu)}\right]\delta_\lambda \ln f'(R) = \\
& = -\frac{1}{2f'(R)}\left[g^{\alpha\epsilon}g_{\beta\mu} - 2\delta^\epsilon_{(\beta}\delta^\alpha_{\mu)}\right[\left[\kappa a_\epsilon + \delta_\epsilon f'(R)\right]\,, \qquad (11)
\end{aligned}
$$

By substituting into the third field equation we obtain

$$
\begin{aligned}
& \overset{g}{\nabla}_\mu\left(gg^{\mu\nu}\right) + g\left(H^\mu_{\lambda\mu}g^{\lambda\nu} + H^\nu_{\lambda\mu}g^{\mu\lambda} - 2H^\lambda_{\lambda\mu}g^{\mu\nu}\right) = 0\,, \\
& \Rightarrow H^\nu_{\lambda\mu}h^{\mu\lambda} - H^\lambda_{\lambda\mu}h^{\mu\nu} = 0\,, \\
& \Rightarrow -\frac{1}{2f'(R)}\left[\left(h^{\nu\epsilon}h_{\lambda\mu} - 2\delta^\epsilon_{(\lambda}\delta^\nu_{\mu)}\right)h^{\mu\epsilon} - \right. \\
& \left. + \left(h^{\lambda\epsilon}h_{\lambda\mu} - 2\delta^\epsilon_{(\lambda}\delta^\lambda_{\mu)}\right)h^{\mu\nu}\right]\left(\kappa a_\epsilon + \delta_\epsilon f'(R)\right) = 0\,, \\
& \Rightarrow -\frac{3}{f'(R)}h^{\nu\epsilon}\left(\kappa a_\epsilon + \delta_\epsilon f'(R)\right) = 0\,, \\
& \Rightarrow a_\epsilon = -\frac{1}{\kappa}\delta_\epsilon f'(R)\,,
\end{aligned} \qquad (12)
$$

where $\overset{g}{\nabla}_\mu$ is now the covariant derivative with respect to the metric g. Hence the matter field $A_\epsilon = \sqrt{g}a_\epsilon = -\frac{\sqrt{g}}{\kappa}\delta_\epsilon f'(R)$ has no dynamics and it is completely determined in terms of the other fields.

We can also express the connection as a function of g alone (or, equivalently, of h alone)

$$
\Gamma^\alpha_{\beta\mu} := \{h\}^\alpha_{\beta\mu} + \frac{1}{2}\left(h^{\alpha\epsilon}h_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right)\delta_\epsilon \ln f'(R) \equiv \{g\}^\alpha_{\beta\mu} \qquad (13)
$$

This behaviour, which has been introduced by the matter coupling, is quite peculiar; the model resembles in the action an $f(R)$ theory but in solution space the connection is directly determined by the original metric rather than by the conformal metric h as in $f(R)$ theories. Still the metric g obeys modified Einstein equations. In fact, we have the first field equation which is now depending on g alone, since the matter and the connection have been determined as functions of g. The *master equation* is obtained as usual by tracing (using $g^{\alpha\beta}$)

$$
f'(R)R - 2f(R) = \kappa T \qquad \Rightarrow f(R) = \frac{1}{2}\left(f'(R)R - \kappa T\right)\,, \qquad (14)
$$

where we set $T := T_{\alpha\beta}g^{\alpha\beta}$ the trace of the stress-energy tensor. By substituting back into the first field equation, being $T = -\frac{3}{\kappa}\Box f'(R)$, we obtain

$$
\begin{aligned}
& f'(R)\left[R_{\alpha\beta} - \frac{1}{4}Rg_{\alpha\beta}\right] - \frac{3}{4}\Box f'(R)g_{\alpha\beta} = \\
& = \nabla_{(\alpha}\nabla_{\beta)}f'(R) - \Box f'(R)g_{\alpha\beta}\,, \\
& \Rightarrow R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \\
& \frac{1}{f'(R)}\left[\nabla_{(\alpha}\nabla_{\beta)}f'(R) - \frac{1}{4}\left(\Box f'(R) + f'R\right)g_{\alpha\beta}\right], \qquad (15)
\end{aligned}
$$

where now the curvature and covariant derivatives refer to g. These are exactly the field equations obtained in the corresponding purely-metric $f(R)$ theory [5].

Hence we have that, regardless of the function $f(R)$, when there is no matter field other than the field A all these models behave exactly as metric $f(R)$ theories. The conformal factor is $f'(R)$ and R can be calculated in terms of T, *i.e.* $\phi(T) = f'(R(T))$. Field equations then imply that Einstein equations hold for the new metric $\hat{g}$ (corresponding to the above h), with a suitably modified stress-energy tensor that takes into account extra effects due to the conformal factor. The previous Einstein equations are recovered

$$\hat{R}_{\alpha\beta} - \frac{1}{2}\hat{R}\,\hat{g}_{\alpha\beta} = \kappa\,\hat{T}_{\alpha\beta} \tag{16}$$

with

$$\hat{T}_{\alpha\beta} = \frac{1}{f'}\left[T_{\alpha\beta} + T^{(grav)}_{\alpha\beta}\right]\,, \tag{17}$$

where the first term on the *rhs* is due to a standard matter term- Clearly Eq.(17) means that the extra degrees of freedom coming from $f(R)$ gravity can be managed as a further contribution to the stress-energy tensor and the above observational shortcomings, related to GR (e.g. DM and DE), can be, in principle, solved in a geometrical way.

Which are the physical implications from the EPS formalism point of view?

1. being g and $\hat{g}$ conformally related photon propagation does not change;

2. Einstein equations hold for the new metric $\hat{g}$ with extra stress-energy tensor directly generated by "ordinary" matter T;

3. rods and clocks change pointwise, by a factor depending on T.

In summary, EPS formalism works also for ETG and further information can be always enclosed in a suitable definition of stress-energy tensor.

VI. A NEW PARADIGM FOR GRAVITY

The coupling of a non-linear gravitational Lagrangian $f(R)$ with matter Lagrangians, depending on the metric g in an arbitrary way or even on the connection Γ in a peculiar way (dictated by EPS compatibility), generate a set of modified Einstein equations in which the following effects are easily recognizable:

A new metric $\hat{g}$ - conformally related to the original metric g - arises. The conformal factor is a computable function of curvature and, through functional inversion, of the trace of the stress tensor that corresponds to the "ordinary" matter distribution (including possible DM and DE effects). In the Palatini approach, the new metric generates the connection Γ as its Levi-Civita connection, so that it describes the free fall of ordinary matter. This new metric induces, in fact, a change of rulers and clocks that affects measurements and conservation laws, while the original g is directly related to light propagation. Due to conformal equivalence, light propagates on the same null geodesics of both g and $\hat{g}$, although clock rates are different in presence of matter.

The net effect of non-linearity and of (non trivial) interaction with matter resides in a change of the stress tensor that couples to the Einstein tensor of $\hat{g}$; a change that induces additions to the previously existing one (directly generated from the matter Lagrangian as discussed above).

This new stress-energy tensor defines conservation laws that are fully covariant with respect to the Einstein frame of $\hat{g}$. Furthermore, it contains an additional term, that can be interpreted under the form of a "space-time varying cosmological constant" $\Lambda(x)$ - in turn determined by distribution of ordinary (and Dark) Matter - so that the residual amount could be interpreted as a net curvature effect (DE) due to the change of rules and clocks induced by EPS compatibility [5]. In other words, the observational effects of such a dynamics are the clustering of astrophysical structures (DM) and the revealed cosmic speed up (DE).

VII. CONCLUSIONS AND REMARKS

To conclude, we can say that very likely Einstein today, after the new phenomenological evidences would much probably come back onto his own steps and accept, as he always did, that models are not eternal and should be dictated by phenomenology rather than by preestablished rules and prejudices. Why should we insist on pre-judicial rules that impose metricity *a priori* (and metricity with respect to a given metric!) and insist on the choice of the "simplest" Hilbert-Lagrangian, when cosmology, quantum Issues and strings suggest instead to us to strictly follow the beautiful analysis of EPS, and work at least *a priori*, in the extended framework of Palatini-EPS formalism and in a much larger class of Lagrangians?

Moreover, let us remark that working in the extended setting suggested by the Palatini-EPS framework requires to reconsider all the machinery and settings of the observational paradigms and protocols have to be carefully analyzed to disentangle purely metrical effects from effects that measure the interaction with free-fall (and therefore with the connection) that in purely metric formalism GR are necessarily mixed up and entangled by the *a priori* requirement that free-fall is also driven by the metric.

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