Effects of Modified Dispersion Relations and Noncommutative Geometry on the Cosmological Constant Computation

Remo Garattini

Università degli Studi di Bergamo, Facoltà di Ingegneria,
Viale Marconi 5, 24044 Dalmine (Bergamo) Italy and
I.N.F.N. - sezione di Milano, Milan, Italy.
E-mail:remo.garattini@unibg.it

Abstract. We compute Zero Point Energy in a spherically symmetric background with the help of the Wheeler-DeWitt equation. This last one is regarded as a Sturm-Liouville problem with the cosmological constant considered as the associated eigenvalue. The graviton contribution, at one loop is extracted with the help of a variational approach together with Gaussian trial functionals. The divergences handled with a zeta function regularization are compared with the results obtained using a Noncommutative Geometry (NCG) and Modified Dispersion Relations (MDR). In both NCG and MDR no renormalization scheme is necessary to remove infinities in contrast to what happens in conventional approaches. Effects on photon propagation are briefly discussed.

Keywords: Cosmological Constant, Quantum Cosmology, Quantum Gravity, Noncommutative Geometry, Modified Dispersion Relations
PACS: 04.60.-m, 04.62+v, 05.10.Cc

INTRODUCTION

The Cosmological Constant problem is certainly one of the most fascinating challenges of our days. A challenge because all the attempts that try to explain the 10^{120} orders of magnitude of discrepancy between the theory and observation have produced unsatisfying results. If we believe that Quantum Field Theory (QFT) is a part of the real world, then the theoretical predictive power to compute the Cosmological Constant must be entrusted to the methods of QFT at the Planck scale. Indeed, calculating the Zero Point Energy (ZPE) of some field of mass m with a cutoff at the Planck scale, we obtain

$$E_{ZPE} = \frac{1}{2}\int_0^{\Lambda_p} \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2} \simeq \frac{\Lambda_p^4}{16\pi^2} \approx 10^{71}GeV^4, \tag{1}$$

while the observation leads to a ZPE of the order $10^{-47}GeV^4$. This surely represents one of the worst predictions of QFT. However if one insists to use the methods of QFT applied to General Relativity, one necessarily meets one of the most famous equations appeared in the literature of Cosmology and Gravity: the Wheeler-DeWitt (WDW) equation[1]. The WDW equation was originally introduced by Bryce DeWitt as an attempt to quantize General Relativity in a Hamiltonian formulation. It is described by

$$\mathscr{H}\Psi = \left[(2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} \left({}^3R - 2\Lambda \right) \right] \Psi = 0 \tag{2}$$

and it represents the quantum version of the classical constraint which guarantees the invariance under time reparametrization. G_{ijkl} is the super-metric, π^{ij} is the super-momentum,${}^3R$ is the scalar curvature in three dimensions and Λ is the cosmological constant, while $\kappa = 8\pi G$ with G the Newton's constant. An immediate application of the WDW Equation is given in terms of the Friedmann-Robertson-Walker (FRW) mini superspace, where all the degrees of freedom but the scale factor are frozen. The FRW metric is described by the following line element

$$ds^2 = -N^2 dt^2 + a^2(t)\, d\Omega_3^2, \tag{3}$$

where $d\Omega_3^2$ is the usual line element on the three sphere, namely

$$d\Omega_3^2 = \gamma_{ij} dx^i dx^j. \tag{4}$$

In this background, we have simply

$$R_{ij} = \frac{2}{a^2(t)} \gamma_{ij} \qquad \text{and} \qquad R = \frac{6}{a^2(t)} \tag{5}$$

and the WDW equation $H\Psi(a) = 0$, becomes

$$\left[-a^{-q} \left[\frac{\partial}{\partial a} a^q \frac{\partial}{\partial a} \right] + \frac{9\pi^2}{4G^2} \left(a^2 - \frac{\Lambda}{3} a^4 \right) \right] \Psi(a) = 0. \tag{6}$$

Eq.(6) assumes the familiar form of a one-dimensional Schrödinger equation for a particle moving in the potential

$$U(a) = \frac{9\pi^2}{4G^2} a^2 \left(1 - \frac{a^2}{a_0^2} \right) \tag{7}$$

with total zero energy. The parameter q represents the factor-ordering ambiguity and $a_0 = \sqrt{\frac{3}{\Lambda}}$ is a reference length. The WDW equation (6) has been solved exactly in terms of Airy functions by Vilenkin[2] for the special case of operator ordering $q = -1$. The Cosmological Constant Λ here appears as a parameter. Nevertheless, except the FRW case and other few examples, the WDW equation is very difficult to solve. This difficulty increases considerably when the mini superspace approach is avoided. However some information can be gained if one changes the point of view. Indeed, instead of treating Λ in Eq.(2) as a parameter, one can formally rewrite the WDW equation as an expectation value computation[1][3]. Indeed, if we multiply Eq.(2) by

---

[1] See also Ref.[4] for an application of the method to a $f(R)$ theory.

$\Psi^*\left[g_{ij}\right]$ and functionally integrate over the three spatial metric g_{ij} we find

$$\frac{1}{V}\frac{\int \mathscr{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi\left[g_{ij}\right]}{\int \mathscr{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\Psi\left[g_{ij}\right]}=\frac{1}{V}\frac{\left\langle\Psi\left|\int_\Sigma d^3x\hat{\Lambda}_\Sigma\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle}=-\frac{\Lambda}{\kappa}. \tag{8}$$

In Eq.(8) we have also integrated over the hypersurface Σ and we have defined

$$V=\int_\Sigma d^3x\sqrt{g} \tag{9}$$

as the volume of the hypersurface Σ with

$$\hat{\Lambda}_\Sigma=(2\kappa)\,G_{ijkl}\pi^{ij}\pi^{kl}-\sqrt{g}^3R/(2\kappa)\,. \tag{10}$$

In this form, Eq.(8) can be used to compute ZPE provided that Λ/κ be considered as an eigenvalue of $\hat{\Lambda}_\Sigma$. In particular, Eq.(8) represents the Sturm-Liouville problem associated with the cosmological constant. To solve Eq.(8) is a quite impossible task. Therefore, we are oriented to use a variational approach with trial wave functionals. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that ZPE should be described by a good candidate of the "*vacuum state*". In the next section we give the general guidelines in ordinary gravity and in presence of Modified Dispersion Relations and the Non Commutative approach to QFT. Units in which $\hbar=c=k=1$ are used throughout the paper.

HIGH ENERGY GRAVITY MODIFICATION: THE EXAMPLE OF NON COMMUTATIVE THEORIES AND GRAVITY'S RAINBOW

As an application of the Eq.(8), we consider the simple example of the Mini Superspace described by a FRW cosmology (3). We find the following simple expectation value

$$\frac{\int \mathscr{D}a\Psi^*(a)\left[-\frac{\partial^2}{\partial a^2}+\frac{9\pi^2}{4G^2}a^2\right]\Psi(a)}{\int \mathscr{D}a\Psi^*(a)\left[a^4\right]\Psi(a)}=\frac{3\Lambda\pi^2}{4G^2}, \tag{11}$$

where the normalization is modified by a weight factor. The application of a variational procedure with a trial wave functional of the form

$$\Psi=\exp\left(-\beta a^2\right) \tag{12}$$

shows that there is no real solution of the parameter β compatible with the procedure. Nevertheless, a couple of imaginary solutions of the variational parameter

$$\beta=\pm i\frac{3\pi}{4G} \tag{13}$$

can be found. These solutions could be related to the tunneling wave function of Vilenkin[2]. Even if the eigenvalue procedure of Eq.(11) leads to an imaginary cosmological constant, which does not correspond to the measured observable, it does not mean that the procedure is useless. Indeed, Eq.(8) can be used to calculate the cosmological constant induced by quantum fluctuations of the gravitational field. To fix ideas, we choose the following form of the metric

$$ds^2 = -N^2(r)\,dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{14}$$

where $b(r)$ is subject to the only condition $b(r_t) = r_t$. We consider $g_{ij} = \bar{g}_{ij} + h_{ij}$,where $\bar{g}_{ij}$ is the background metric and h_{ij} is a quantum fluctuation around the background. Then we expand Eq.(8) in terms of h_{ij}. Since the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g}\,{}^3R$ up to second order in h_{ij}[2]. As shown in Ref.[6], the final contribution does not include ghosts and simply becomes

$$\frac{1}{V}\frac{\left\langle \Psi^\perp \left| \int_\Sigma d^3x \left[\hat{\Lambda}_\Sigma^\perp\right]^{(2)} \right| \Psi^\perp \right\rangle}{\left\langle \Psi^\perp | \Psi^\perp \right\rangle} = -\frac{\Lambda^\perp}{\kappa}. \tag{15}$$

The integration over Gaussian Wave functionals leads to

$$\hat{\Lambda}_\Sigma^\perp = \frac{1}{4V}\int_\Sigma d^3x\sqrt{\bar{g}}G^{ijkl}\left[(2\kappa)K^{-1\perp}(x,x)_{ijkl} + \frac{1}{(2\kappa)}\left(\tilde{\triangle}_L\right)^a_j K^\perp(x,x)_{iakl}\right], \tag{16}$$

where

$$\left(\tilde{\triangle}_L h^\perp\right)_{ij} = \left(\triangle_L h^\perp\right)_{ij} - 4R^k_i h^\perp_{kj} + {}^3R h^\perp_{ij} \tag{17}$$

is the modified Lichnerowicz operator and $\triangle_L$is the Lichnerowicz operator defined by

$$(\triangle_L h)_{ij} = \triangle h_{ij} - 2R_{ikjl}h^{kl} + R_{ik}h^k_j + R_{jk}h^k_i \qquad \triangle = -\nabla^a\nabla_a. \tag{18}$$

G^{ijkl} represents the inverse DeWitt metric and all indices run from one to three. Note that the term $-4R^k_i h^\perp_{kj} + {}^3R h^\perp_{ij}$ disappears in four dimensions. The propagator $K^\perp(x,x)_{iakl}$ can be represented as

$$K^\perp(\overrightarrow{x},\overrightarrow{y})_{iakl} = \sum_\tau \frac{h^{(\tau)\perp}_{ia}(\overrightarrow{x})\, h^{(\tau)\perp}_{kl}(\overrightarrow{y})}{2\lambda(\tau)}, \tag{19}$$

where $h^{(\tau)\perp}_{ia}(\overrightarrow{x})$ are the eigenfunctions of $\tilde{\triangle}_L$. τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq.(16). The expectation value of $\hat{\Lambda}_\Sigma^\perp$ is easily obtained by inserting the form of the propagator

[2] See Refs.[4, 5, 6] for technical details.

into Eq.(16) and minimizing with respect to the variational function $\lambda(\tau)$. Thus the total one loop energy density for TT tensors becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2}\sum_{\tau}\left[\sqrt{\omega_1^2(\tau)}+\sqrt{\omega_2^2(\tau)}\right]. \tag{20}$$

The above expression makes sense only for $\omega_i^2(\tau) > 0$, where ω_i are the eigenvalues of $\tilde{\triangle}_L$. Following Refs.[4, 5, 6], we find that the final evaluation of expression (20) is

$$\frac{\Lambda}{8\pi G} = -\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{+\infty}\omega_i \frac{d\tilde{g}(\omega_i)}{d\omega_i}d\omega_i = -\frac{1}{4\pi^2}\sum_{i=1}^{2}\int_{\sqrt{m_i^2(r)}}^{+\infty}\omega_i^2\sqrt{\omega_i^2-m_i^2(r)}d\omega_i, \tag{21}$$

where we have included an additional 4π coming from the angular integration and where we have defined two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2}\left(1-\frac{b(r)}{r}\right)+\frac{3}{2r^2}b'(r)-\frac{3}{2r^3}b(r) \\ \\ m_2^2(r) = \frac{6}{r^2}\left(1-\frac{b(r)}{r}\right)+\frac{1}{2r^2}b'(r)+\frac{3}{2r^3}b(r) \end{cases} \quad (r\equiv r(x)). \tag{22}$$

The effective masses have different expression from case to case. For example, in the Schwarzschild case, $b(r) = 2MG$, we find

$$\begin{cases} m_1^2(r) = \frac{6}{r^2}\left(1-\frac{2MG}{r}\right)-\frac{3MG}{r^3} \\ \\ m_2^2(r) = \frac{6}{r^2}\left(1-\frac{2MG}{r}\right)+\frac{3MG}{r^3} \end{cases} \quad (r\equiv r(x)). \tag{23}$$

The expression in Eq.(21) is divergent and must be regularized. For example, the zeta function regularization method leads to

$$\rho_i(\varepsilon) = \frac{m_i^4(r)}{64\pi^2}\left[\frac{1}{\varepsilon}+\ln\left(\frac{4\mu^2}{m_i^2(r)\sqrt{e}}\right)\right] \quad i=1,2 \quad , \tag{24}$$

where an additional mass parameter μ has been introduced in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. The renormalization is performed via the absorption of the divergent part into the re-definition of a bare classical quantity and the final result is given by

$$\frac{\Lambda_0}{8\pi G} = \frac{m_1^4(r)}{64\pi^2}\ln\left(\frac{4\mu^2}{m_1^2(r)\sqrt{e}}\right)+\frac{m_2^4(r)}{64\pi^2}\ln\left(\frac{4\mu^2}{m_2^2(r)\sqrt{e}}\right). \tag{25}$$

Of course, one can follow other methods to obtain finite results: for instance, the use of a UV-cut off. Nevertheless it is possible to obtain finite results introducing a distortion in the space-time from the beginning. This can be realized with the help of the Non Commutative Approach to QFT developed in Ref.[7] or with the help of Gravity's

Rainbow developed in Ref.[9]. Noncommutative theories provide a powerful method to naturally regularize divergent integrals appearing in QFT. Eq.(21) is a typical example of a divergent integral. The noncommutativity of spacetime is encoded in the commutator $[\mathbf{x}^{\mu},\mathbf{x}^{\nu}]=i\,\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix which determines the fundamental discretization of spacetime. In even dimensional space-time, $\theta^{\mu\nu}$ can be brought to a block-diagonal form by a suitable Lorentz rotation leading to

$$\theta^{\mu\nu}=diag\left[\theta_1\varepsilon^{ab}\theta_2\varepsilon^{ab}\ldots\theta_{d/2}\varepsilon^{ab}\right] \tag{26}$$

with ε^{ab} a 2×2 antisymmetric Ricci Levi-Civita tensor. If $\theta_i\equiv\theta\ \forall i=1\ldots d/2$, then the space-time is homogeneous and preserves isotropy. The effect of the θ length on ZPE calculation is basically the following: the classical Liouville counting number of nodes

$$dn=\frac{d^3\vec{x}d^3\vec{k}}{(2\pi)^3}, \tag{27}$$

is modified by distorting the counting of nodes in the following way[7]

$$dn=\frac{d^3xd^3k}{(2\pi)^3}\implies dn_i=\frac{d^3xd^3k}{(2\pi)^3}\exp\left(-\frac{\theta}{4}\left(\omega_{i,nl}^2-m_i^2(r)\right)\right),\quad i=1,2. \tag{28}$$

This deformation corresponds to an effective cut off on the background geometry (14). The UV cut off is triggered only by higher momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. As an effect the final induced cosmological constant becomes

$$\frac{\Lambda}{8\pi G}=\frac{1}{6\pi^2}\left[\int_{\sqrt{m_1^2(r)}}^{+\infty}\sqrt{\left(\omega^2-m_1^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2-m_1^2(r)\right)}d\omega\right.$$
$$\left.+\int_{\sqrt{m_2^2(r)}}^{+\infty}\sqrt{\left(\omega^2-m_2^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2-m_2^2(r)\right)}d\omega\right], \tag{29}$$

where an integration by parts in Eq.(21) has been done. We recover the usual *divergent* integral when $\theta\to 0$. The result is finite and we have an induced cosmological constant which is regular. We can obtain enough information in the asymptotic régimes when the background satisfies the relation

$$m_0^2(r)=m_1^2(r)=-m_2^2(r), \tag{30}$$

which is valid for the Schwarzschild, Schwarzschild-de Sitter (SdS) and Schwarzschild-Anti de Sitter (SAdS) metric close to the throat. Indeed, defining

$$x=\frac{m_0^2(r)\,\theta}{4}, \tag{31}$$

we find that when $x\to+\infty$,

$$\frac{\Lambda}{8\pi G}\simeq\frac{1}{6\pi^2\theta^2}\sqrt{\frac{\pi}{x}}\left[3+\left(8x^2+6x+3\right)\exp(-x)\right]\to 0. \tag{32}$$

Conversely, when $x \to 0$, we obtain

$$\frac{\Lambda}{8\pi G} \simeq \frac{4}{3\pi^2\theta^2}\left[2-\left(\frac{7}{8}+\frac{3}{4}\ln\left(\frac{x}{4}\right)+\frac{3}{4}\gamma\right)x^2\right] \to \frac{8}{3\pi^2\theta^2}. \tag{33}$$

The other interesting cases, namely de Sitter and Anti-de Sitter and Minkowski are described by

$$m_1^2(r) = m_2^2(r) = m_0^2(r), \tag{34}$$

leading to

$$\frac{\Lambda}{8\pi G} \simeq \frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2 \frac{3}{8}\sqrt{\frac{\pi}{x}} \to 0, \tag{35}$$

when $x \to \infty$ and

$$\frac{\Lambda}{8\pi G} \simeq \frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2\left[1-\frac{x}{2}+\left(-\frac{7}{16}-\frac{3}{8}\ln\left(\frac{x}{4}\right)-\frac{3}{8}\gamma\right)x^2\right] \to \frac{8}{3\pi^2\theta^2}, \tag{36}$$

when $x \to 0$. As regards Gravity's Rainbow[8], we can begin by defining a"*rainbow metric*"

$$ds^2 = -\frac{N^2(r)\,dt^2}{g_1^2(E/E_P)} + \frac{dr^2}{\left(1-\frac{b(r)}{r}\right)g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)}\left(d\theta^2+\sin^2\theta\, d\phi^2\right). \tag{37}$$

$g_1(E/E_P)$ and $g_2(E/E_P)$ are two arbitrary functions which have the following property

$$\lim_{E/E_P\to 0} g_1(E/E_P) = 1 \qquad \text{and} \qquad \lim_{E/E_P\to 0} g_2(E/E_P) = 1. \tag{38}$$

We expect the functions $g_1(E/E_P)$ and $g_2(E/E_P)$ modify the UV behavior in the same way as GUP and Noncommutative geometry do, respectively. Following Ref.[9], in presence of Gravity's Rainbow, we find that Eq.(8) changes into

$$\frac{g_2^3(E/E_P)}{\tilde{V}}\frac{\left\langle \Psi \left| \int_\Sigma d^3x \tilde{\Lambda}_\Sigma \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}, \tag{39}$$

where

$$\tilde{\Lambda}_\Sigma = (2\kappa)\frac{g_1^2(E/E_P)}{g_2^3(E/E_P)}\tilde{G}_{ijkl}\tilde{\pi}^{ij}\tilde{\pi}^{kl} - \frac{\sqrt{\tilde{g}}\tilde{R}}{(2\kappa)\,g_2(E/E_P)}. \tag{40}$$

Of course, Eq.(39) and Eq.(40) reduce to the ordinary Eqs.$(2,8)$ and (10) when $E/E_P \to 0$. By repeating the procedure leading to Eq.(20), we find that the TT tensor contribution of Eq.(39) to the total one loop energy density becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2}\sum_{i=1}^{2}\int_{E^*}^{+\infty} E_i g_1(E/E_P)\, g_2(E/E_P)\frac{d}{dE_i}\sqrt{\left(\frac{E_i^2}{g_2^2(E/E_P)} - m_i^2(r)\right)^3}\, dE_i, \tag{41}$$

where E^* is the value which annihilates the argument of the root. In the previous equation we have assumed that the effective mass does not depend on the energy E. To further proceed, we choose a form of $g_1(E/E_P)$ and $g_2(E/E_P)$ which allows a comparison with the results obtained with a Noncommutative geometry computation expressed by Eq.(29). We are thus led to choose

$$g_1(E/E_P) = \left(1+\beta\frac{E}{E_P}\right)\exp(-\alpha\frac{E^2}{E_P^2}) \qquad \text{and} \qquad g_2(E/E_P) = 1, \tag{42}$$

with $\alpha > 0$ and $\beta \in \mathbb{R}$, because the pure "*Gaussian*" choice with $\beta = 0$ can not give a positive induced cosmological constant[3]. However this is true when the effective masses satisfy relation (30). In case relation (34) holds the pure "*Gaussian*" choice works for large and small x, where $x = \sqrt{m_0^2(r)/E_P^2}$. The final result is a vanishing induced cosmological constant in both asymptotic régimes. It is interesting to note that Gravity's Rainbow has potential effects on the photon propagation[10]. Indeed, let us consider two photons emitted at the same time $t = -t_0$ at $x_{dS} = 0$. The first photon be a low energy photon ($E \ll E_P$) and the second one be a Planckian photon ($E \sim E_P$). Both photons are assumed to be detected at a later time in $\bar{x}_{dS}$. We expect to detect the two photons with a time delay Δt given by the solution of the equation

$$\bar{x}_{dS}^{E\ll E_P}(0) = \bar{x}_{dS}^{E\sim E_P}(\Delta t), \tag{43}$$

that implies

$$\Delta t \simeq g_1(E)\frac{e^{\sqrt{\Lambda_{eff}/3}t_0} - e^{\frac{\sqrt{\Lambda_{eff}/3}}{g_1(E)}t_0}}{\sqrt{\Lambda_{eff}/3}} \simeq \beta\frac{E}{E_P}t_0\left(1+\sqrt{\Lambda_{eff}/3}t_0\right), \tag{44}$$

where we have used (42) for the rainbow functions.

REFERENCES

1. B. S. DeWitt, *Phys. Rev.* **160**, 1113 (1967).
2. A. Vilenkin, *Phys. Rev.* **D 37**, 888 (1988).
3. R.Garattini, *J. Phys. A* **39**, 6393 (2006); gr-qc/0510061. R. Garattini, *J.Phys.Conf.Ser.* **33**, 215 (2006); gr-qc/0510062.
4. S. Capozziello and R. Garattini, *Class.Quant.Grav.* **24**, 1627 (2007); gr-qc/0702075.
5. R. Garattini, *TSPU Vestnik* **44 N7**, 72 (2004); gr-qc/0409016.
6. R. Garattini, *The Cosmological constant and the Wheeler-DeWitt Equation*, PoS CLAQG08 (2011) 012; arXiv:0910.1735 [gr-qc];
7. R. Garattini and P. Nicolini, *Phys. Rev.* **D 83**, 064021 (2011); arXiv:1006.5418 [gr-qc].
8. J. Magueijo and L. Smolin, *Class. Quant. Grav.* **21**, 1725 (2004) [arXiv:gr-qc/0305055].
9. R. Garattini and G. Mandanici, *Phys. Rev.* **D 83**, 084021 (2011), arXiv:1102.3803 [gr-qc].
10. R. Garattini and G. Mandanici,*Particle propagation and effective space-time in Gravity's Rainbow.* To appear in Phys.Rev.D; e-Print: arXiv:1109.6563 [gr-qc].

[3] See Ref.[9] for technical details.