Fuzzy Space-time, Quantization and Gauge Invariance

S.N.Mayburov
Lebedev Inst. of Physics
Leninsky Prospect 53, Moscow, Russia, RU-117924
e-mail: mayburov@sci.lebedev.ru

Abstract

Quantum space-time with Dodson-Zeeman topological structure is studied. In its framework the states of massive particle m correspond to elements of fuzzy set called fuzzy points. Due to their weak ordering, m space coordinate x acquires principal uncertainty σ_x. Quantization formalism is derived from consideration of m evolution in fuzzy phase space with minimal number of additional assumptions. Particle's interactions on fuzzy manifold are studied and shown to be gauge invariant.

Structure of space-time at microscopic (Plank) scale and its relation to axiomatic of Quantum Mechanics (QM) is actively discussed now [1, 2]. In particular, it was proposed that such fundamental properties of space-time manifold M_{ST} as its metrics and topology can differ significantly at Planck scale from standard Riemanian formalism [2, 3]. In particular, Posets and the fuzzy ordered sets (Fosets) were used for the construction of different variants of the novel Fuzzy Topology (FT) [4, 5]. Hence it's instructive to study what kind of physical theory such topology induces [1, 3]. In our previous works it was shown that in its framework the quantization procedure by itself can be defined as the transition from the ordered phase space to fuzzy one. Therefore, the quantum properties of particles and fields are induced directly by FT of their phase space and don't need to be postulated separately of it [1, 3]. As the simple example of such transition the quantization of nonrelativistic particle was regarded; it was argued that FT induces the particle's dynamics which is equivalent to QM evolution [1, 3]. Yet in its derivation some phenomenological assumptions were used, here the new and rather simple formalism, which permits to drop them, will be described. It will be shown also that the interactions on such fuzzy manifold are gauge invariant and under simple assumptions correspond to Yang-Mills fields [3]. It is worth to mention here the extensive studies of noncommutative fuzzy spaces, both finite (sphere, tori) and infinite ones, these are, in fact, the alternative and more complicated variants of such formalism [6].

Here we consider only the most important steps in construction of mechanics on fuzzy manifold called fuzzy mechanics (FM), the details can be found in [1, 3] . In 1-dimensional Euclidean Geometry, the elements of its manifold X are the points x_a which constitute the ordered set. For the elements of partially ordered set (Poset) $\{d_i\}$, beside the standard ordering relation between its elements $d_k \leq d_l$ (or vice versa), the incomparability relation $d_k \wr d_l$ is also permitted; if it's true, then both $d_k \leq d_l$ and $d_l \leq d_k$ propositions are false. To illustrate its meaning, consider Poset $D^T = A \cup B$, which includes the subset of 'incomparable' elements $B = \{b_j\}$, and the ordered subset $A = \{a_i\}$. Let's suppose that in A the element's indexes grow correspondingly to their ordering, so that $\forall\, i,\ a_i \leq a_{i+1}$. As the example, consider some interval $\{a_l, a_{l+n}\}$ and suppose that $b_j \in \{a_l, a_{l+n}\}$, i.e. $a_l \leq b_j$; $b_j \leq a_{l+n}$ and $b_j \wr a_i$; iff $l+1 \leq i \leq l+n-1$. In this case, b_j in some sense is 'smeared' over $\{a_l, a_{l+n}\}$ interval. To introduce the fuzzy relations, let's put in correspondence to each b_j, a_i pair the weight $w_i^j \geq 0$ with the norm $\sum_i w_i^j = 1$. Under this conditions D^T is Foset, b_j called the fuzzy points. The continuous 1-dimensional Foset C^F is defined analogously; $C^F = B \cup X$ where B is the same as above, X is the continuous ordered subset, which is equivalent to R^1 axis of real numbers. Correspondingly, fuzzy relation between b_j, x_a are described by $w^j(x_a) \geq 0$ with the norm $\int w^j dx_a = 1$. Note that in Fuzzy Topology $w^j(x)$ doesn't have any probabilistic meaning but only the algebraic one.

In these terms the particle's state in 1-dimensional classical mechanics corresponds to ordered point $x(t)$ in X. Analogously to it, in 1-dimensional Fuzzy Mechanics (FM) the particle m corresponds to fuzzy point $b(t)$ in C^F; it characterized by normalized positive density $w(x,t)$. However, m fuzzy state $|g\}$ can depend on other m degrees of freedom (DF). The obvious one is $\frac{\partial w}{\partial t}$, yet it's more convenient to replace it by related DF, which describes w flow velocity $v(x,t)$. Assuming FM locality, flow continuity equation should hold for w flow $\vec{j}$:

$$\frac{\partial w}{\partial t}(x) = -div\vec{j} = -v\frac{\partial w}{\partial x} - \frac{\partial v}{\partial x}w \tag{1}$$

Really, its violation would correspond to long-distance w correlations incompatible with FT. Below $v(x)$ will be replaced by hydromechanical velocity potential:

$$\gamma(x) = r\int\limits_{-\infty}^{x} v(\xi)d\xi \tag{2}$$

where r is an arbitrary constant. If $|g\}$ doesn't depend on any other DFs, then analogously to dirac vector in $X-$representation it can be unambiguously expressed as:

$$g(x) = \sqrt{w(x)}e^{i\gamma(x)}. \tag{3}$$

Evolution equation for m free motion should be of the first order in time and so it can be written as:

$$i\frac{\partial g}{\partial t} = \hat{H}g. \tag{4}$$

In general $\hat{H}$ is nonlinear operator, for the simplicity we shall consider first the linear case and turn to nonlinear one afterwards. The free m evolution is invariant relative to X shifts performed by the operator $\hat{W}(a) = \exp(a\frac{\partial}{\partial x})$. Because of it, $\hat{H}$ should commute with $\hat{W}(a)$ for the arbitrary a, i. e. $[\hat{H}, \frac{\partial}{\partial x}] = 0$. It holds only if $\hat{H}$ is differential polinom, which can be written as:

$$\hat{H} = \hat{H}_0 + \Delta\hat{H} = -c_1\frac{\partial}{\partial x} - c_2\frac{\partial^2}{\partial x^2} - \sum_{l=3}^{n} c_l\frac{\partial^l}{\partial x^l} \tag{5}$$

where $\Delta\hat{H}$ denotes the sum over l; $c_{1,2}, c_l$ are arbitrary constants, $n \geq 3$. If to substitute $v(x)$ by $\gamma(x)$ in eq. (1) and transform it to $\sqrt{w}$ time derivative, then the left part of (4) is equal to:

$$i\frac{\partial g}{\partial t}(x) = -(\frac{i}{r}\frac{\partial\sqrt{w}}{\partial x}\frac{\partial\gamma}{\partial x} + \frac{i}{2r}\sqrt{w}\frac{\partial^2\gamma}{\partial x^2} + \sqrt{w}\frac{\partial\gamma}{\partial t})e^{i\gamma} \tag{6}$$

For $c_1 = 0$ the imaginary terms of (6) and $\hat{H}_0 g$ coincide up to c_2/r ratio, from that n can be obtained. Really, imaginary part of $\Delta\hat{H}g$ should include the term proportional to $c_n\frac{\partial^n\gamma}{\partial x^n}$, yet eq. (6) includes the highest term corresponding to $n = 2$ only. Hence if to settle that $c_2 = \frac{1}{2r}$, then $\Delta H = 0$ and $c_1 = 0$, as the result Schroedinger equation for free particle with mass $m_0 = r$ is obtained for m evolution. Note that in FM it's equivalent to the system of two equations, one for w and other for γ time derivatives, their meaning will be discussed below. Plainly, $\gamma(x)$ corresponds to quantum phase, the states described by $g(x)$ are dirac vectors (rays) of QM Hilbert space $\mathcal{H}$ [8].

Concerning with nonlinear case, the conditions of dynamics linearity were obtained by Jordan, and turn out to be rather weak [7]. In particular, it was shown that if the evolution maps the set of pure states onto itself, then such evolution is linear. Yet for FM such condition is generic, no mixed state can appear in free evolution of fuzzy state, and so FM evolution should be linear [1]. In FM x is m observable and it's sensible to admit that $\hat{p}_x = i\frac{\partial}{\partial x}$ describes m momentum and all operator functions $\hat{F}_Q(x,p)$ are also m observables. Hence in such formalism the commutation relations of the kind $[x, p_x] = i$ are obtained from topological premises which constitute FM basis. Generalization of FM formalism on 3 dimensions is straightforward and doesn't demand any serious modification of described ansatz.

Note that Planck constant $\hbar = 1$ in our FM ansatz, but the same value ascribed to it in Relativistic unit system together with $c = 1$; in FM framework $\hbar$ only connects x, p scales and doesn't have any other meaning. For relativistic free evolution the linearity of state evolution becomes the important criterion for the choice of consistent ansatz. For massive particle m the minimal solution is 4-spinor $g_i(\vec{r}, t)$; $i = 1, 4$, its evolution is described by Dirac equation for spin-$\frac{1}{2}$, i. e. such particle is fermion.

Now we shall consider the interaction between fuzzy states in nonrelativstic FM and attempt to extend the obtained results on relativistic case. Note first that by derivation FM free Hamiltonian H_0 induces $\mathcal{H}$ dynamical asymmetry between $|\vec{r}\rangle$ and $|\vec{p}\rangle$ 'axes' which is absent in standard QM formalism. As was shown, in FM m free dynamics is described by the system of two equations which define $\frac{\partial\sqrt{w}}{\partial t}$ and $\frac{\partial\gamma}{\partial t}$. Yet the first of them is equivalent to eq. (1) which describes $w(x)$ balance and so is, in fact, kinematical one. Thus any m interactions can be accounted only via second equation:

$$\frac{\partial\gamma}{\partial t} = \frac{1}{2m_0}\frac{(\partial\gamma}{\partial x)^2}\frac{1}{\sqrt{w}}\frac{\partial^2\sqrt{w}}{\partial x^2} + H_{int} \tag{7}$$

Since γ corresponds to quantum phase, it supposes that in FM all m interactions should be gauge invariant. Despite that the fermion state is described by several phases the same invariance if fulfilled for them and can be extended also on relativistic case. Basing on it, QED formalism was derived by us with the minimum of additional assumptions. The preliminary results for the interactions of fermion multiplets show that in such theory their interactions also possess $SU(n)$ gauge invariance and transferred by the corresponding Yang-Mills fields [3].

In conclusion, we have shown that the quantization of elementary systems can be derived directly from axiomatic of Set Theory and Topology together with the natural assumptions about system evolution. It allows to suppose that the quantization phenomenon has its roots in foundations of mathematics and logics [8]. The main aim of FM, as well as other studies of fuzzy spaces, is the construction of nonlocal QFT (or other more general theory). In this vein, FM provides the interesting opportunities, being generically nonlocal theory which, in the same time, is Lorentz covariant and manifests the gauge invariance.

References

[1] Mayburov S (2008) Fuzzy Geometry of Phase Space and Quantization of Massive Fields, J.Phys A (41) 164071-164080

[2] Isham C (1994) Introduction into Canonical Gravity Quantization, in Canonical Gravity: from Classical to Quantum , Lecture Notes in Phys., Springer, Berlin, (433) 11-28

[3] Mayburov S (2010) Fuzzy Space-Time Geometry and Particle's Dynamics, Int. J. Theor. Phys. (4), 3192-3198

[4] Zeeman C (1961) Topology of Brain and Visual Perception in Topology of 3-manifolds , Prentice-Hall, New Jersey, 240-248

[5] Dodson C (1975) Tangent Structures for Hazy Space, J. London Math. Soc. (2), 465-474

[6] Madore J (1992) Fuzzy Sphere, Class. Quant. Gravity (9), 69-83

[7] Jordan T (2006) Assumptions of Linear Dynamics, Phys. Rev. (A73), 022101-022108

[8] Jauch J (1968) Foundations of Quantum Mechanics, Addison-Wesly, Reading