On Fluid Maxwell Equations

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Abstract

Fluid mechanics is a field theory of Newtonian mechanics of Galilean symmetry, concerned with fluid flows represented by the velocity field such as $\boldsymbol{v}(\boldsymbol{x}, t)$ in space-time. A fluid is a medium of continuous mass. Its mechanics is formulated by extending discrete system of point masses. Associated with two symmetries (translation and space-rotation), there are two gauge fields: $\boldsymbol{E} \equiv (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and $\boldsymbol{H} \equiv \nabla \times \boldsymbol{v}$, which do not exist in the system of discrete masses. One can show that those are analogous to the electric field and magnetic field in the electromagnetism, and *fluid Maxwell equations* can be formulated for $\boldsymbol{E}$ and $\boldsymbol{H}$. Sound waves within the fluid is analogous to the electromagnetic waves in the sense that phase speeds of both waves are independent of wave lengths, *i.e.* non-dispersive.

1 Introduction

Fluid mechanics is a field theory of Galilean symmetry. Two symmetries are known as subgroups of the Galilean group: translation (space and time) and space-rotation. From this point of view, a gauge-theoretic study has been developed by Kambe (2008) for flows of an ideal compressible fluid with respect to both global and local invariances of the flow fields in the space-time $(\boldsymbol{x},\, t)$, where $\boldsymbol{x} = (x^1, x^2, x^3)$ is the three-dimensional space coordinates.

Suppose that we have a velocity field $\boldsymbol{v}(t, \boldsymbol{x})$ depending on $\boldsymbol{x}$ and time t. Then one can define the convective derivative D_t (*i.e.* the Lagrange derivative in the fluid mechanics) by

$$\mathrm{D}_t \equiv \partial_t + \boldsymbol{v} \cdot \nabla\,, \qquad \text{where } \ \partial_t = \partial/\partial t, \quad \nabla = (\partial/\partial x^i). \tag{1}$$

It is remarkable that D_t is *gauge-invariant*, namely D_t is invariant with respect to local gauge transformations (Kambe 2008). This implies that the gauge theory can be applied to the fluid mechanics, since the covariant derivative is an essential building block of the gauge theory (Kambe 2010a).

The fluid velocity $\boldsymbol{v}$ and acceleration $\mathcal{A}$ are defined by $\boldsymbol{v} = \mathrm{D}_t\boldsymbol{x}$, and $\mathcal{A} = \mathrm{D}_t\boldsymbol{v} \ = \partial_t\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, respectively. The second term on the right of $\mathcal{A}$ can be transformed by the following vector identity:

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) + \nabla(\tfrac{1}{2}\,|\boldsymbol{v}|^2). \tag{2}$$

It is verified by Kambe (2008, 2010a) that the vorticity defined by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \mathrm{curl}\,\boldsymbol{v} \tag{3}$$

is a *gauge field* with respect to the rotation symmetry, and the acceleration $\mathcal{A}$ is given by the following gauge-invariant expression: $\mathrm{D}_t\boldsymbol{v} = \partial_t\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \partial_t\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \nabla(\frac{1}{2}\,|\boldsymbol{v}|^2)$.

In a dynamical system of n discrete point masses with their positions denoted by $\boldsymbol{X}_k(t)$ ($k = 1, \cdots, n$), the equations of motion are described by the form, $(\mathrm{d}/\mathrm{d}t)^2\boldsymbol{X}_k(t) = \boldsymbol{F}_k(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n)$, where $\boldsymbol{F}_k$ is the force acting on the k-th particle. For a fluid of continuous mass distribution, the discrete positions $\boldsymbol{X}_k(t)$ are replaced by a continuous field representation $\boldsymbol{X}(\boldsymbol{a}, t)$, where $\boldsymbol{a}$ is the Lagrange parameters $\boldsymbol{a} = (a_1, a_2, a_3)$, or labels identifying fluid particles. Associated with the field representaton, the time derivative $\mathrm{d}/\mathrm{d}t$ should be replaced by D_t, and the acceleration is given by $\partial_t\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$. Thus, we have a *connection* term $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, which is the second *gauge field*.

Clearly, the two gauge fields $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and $\nabla \times \boldsymbol{v}$ do not exist in discrete systems of point masses. They are the fields defined only in continuous differentiable fields such as the fluid flow. In fact, it is remarkable that they are analogous to the electric field and magnetic field in the electromagnetism, and a system of fluid Maxwell equations can be formulated for $\boldsymbol{E} \equiv (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ and $\boldsymbol{H} \equiv \nabla \times \boldsymbol{v}$ (Kambe 2010b).

It is shown in the next section 2 that there exists a similarity between the wave equations of electromagnetism and fluid mechanics. This suggests existence of correspondence between the variables of electromagnetism and fluid-mechanics. The section 3 describes that the correspondence permits formulation of a system of fluid Maxwell equations. In the section 4 a wave equation is derived from the system for a vector field. However in the fluid case, the vector form reduces to a scalar wave equation for sound waves, since all the terms of the equation are expressed by gradients of scalar fields. In the last section 5, another analogy between fluid mechanics and electromagnetism is presented for the equations of motion of a test particle in flow field and in electromagnetic field.

2 Equations of Fluid mechanics, compared with Electromagnetism

2.1 Equations of Fluid mechanics

Equation of motion of an ideal fluid is given by the Euler's equation of motion:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla\, p. \tag{4}$$

This equation is supplemented by the followings:

$$\begin{aligned} \partial_t \rho + \boldsymbol{v}\cdot\nabla\rho + \rho\,\nabla\cdot\boldsymbol{v} &= 0\,, & (5)\\ \partial_t s + \boldsymbol{v}\cdot\nabla s &= 0\,, & (6)\\ \partial_t \boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) &= 0, & (7) \end{aligned}$$

where ρ is the fluid density, s the entropy per unit mass, p the pressure. The equation (5) is the continuity equation, and (7) is the vorticity equation, while (6) is the entropy equation, stating that each fluid particle keeps its initial entropy (*i.e.* adiabatic).

If initial entropy field is uniform with a constant value s_0, the fluid keeps the *isentropic* state $s = s_0$ at any later time and everywhere. In this case, we have $(1/\rho)\nabla\, p = \nabla\, h$ by the thermodynamics where h is the enthalpy per unit mass.[1] In isentropic flows, an enthalpy variation Δh and a density variation $\Delta\rho$ are related by

$$\Delta h = \frac{1}{\rho}\,\Delta p = \frac{a^2}{\rho}\,\Delta\rho, \qquad \text{where}\ \ \Delta p = a^2\,\Delta\rho, \quad a^2 = (\partial p/\partial\rho)_s. \tag{8}$$

The notation $(\partial p/\partial\rho)_s$ denotes partial differentiation with s fixed, and $a = \sqrt{(\partial p/\partial\rho)_s}$ is the *sound speed*, as becomes clear below. From the above, we have $\partial_t\rho = (\rho/a^2)\partial_t h$ and $\nabla\rho = (\rho/a^2)\nabla h$. Therefore, the equation (5) is transformed to $(\rho/a^2)\left(\partial_t h + \boldsymbol{v}\cdot\nabla h + a^2\nabla\cdot\boldsymbol{v}\right) = 0$. Thus, the fluid equations (4) $\sim$ (7) reduce to the following three equations:

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\, h &= 0, & (9)\\ \partial_t h + \boldsymbol{v}\cdot\nabla h + a^2\nabla\cdot\boldsymbol{v} &= 0, & (10)\\ \partial_t \boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) &= 0. & (11) \end{aligned}$$

Using (2), the equation of motion (9) is rewritten as[2]

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} + \nabla(\tfrac{1}{2}\,v^2 + h) = 0. \tag{12}$$

2.2 Equations of Electromagnetism

In electromagnetism, Maxwell's equations for electric field $\boldsymbol{E}^{\rm em}$ and magnetic field $\boldsymbol{H}^{\rm em}$ are

$$\begin{aligned} \nabla\times\boldsymbol{E}^{\rm em} + c^{-1}\partial_t\boldsymbol{H}^{\rm em} &= 0\,, \qquad & \nabla\cdot\boldsymbol{H}^{\rm em} = 0,\\ \nabla\times\boldsymbol{H}^{\rm em} - c^{-1}\partial_t\boldsymbol{E}^{\rm em} &= \boldsymbol{J}^{\rm e}, \qquad & \nabla\cdot\boldsymbol{E}^{\rm em} = q^{\rm e}, \qquad (13) \end{aligned}$$

[1] From the thermodynamics, d$h = (1/\rho)\,$d$p + T\,$ds where T is the temperature. If d$s = 0$, we have d$h = (1/\rho)\,$dp.

[2] The vorticity equation (11) is also obtained by taking curl of (12).

where $q^{\rm e} = 4\pi\,\rho^{\rm e}$ and $\boldsymbol{J}^{\rm e} = (4\pi/c)\boldsymbol{j}^{\rm e}$ with $\rho^{\rm e}$ and $\boldsymbol{j}^{\rm e}$ being the charge density and current density vector respectively, and c the light velocity. The vector fields $\boldsymbol{E}^{\rm em}$ and $\boldsymbol{H}^{\rm em}$ can be defined in terms of a vector potential $\boldsymbol{A}$ and a scalar potential $\phi^{(\rm e)}$ by

$$\boldsymbol{E}^{\rm em} = -c^{-1}\partial_t \boldsymbol{A} - \nabla\,\phi^{(\rm e)}, \qquad \boldsymbol{H}^{\rm em} = \nabla \times \boldsymbol{A}. \tag{14}$$

Using these definitions, the above Maxwell equations require that the two fields $\boldsymbol{A}$ and $\phi^{(\rm e)}$ satisfy the following equations (Landau & Lifshitz (1975), Chap.8):

$$\partial_t \phi^{(\rm e)} + c\,\nabla\cdot\boldsymbol{A} = 0 \qquad \text{(Lorentz condition)}, \tag{15}$$

$$(\partial_t^2 - c^2\nabla^2)\,\boldsymbol{A} = c^2\boldsymbol{J}^{\rm e}, \qquad (\partial_t^2 - c^2\nabla^2)\,\phi^{(\rm e)} = c^2 q^{\rm e}. \tag{16}$$

2.3 Analogy in wave property

Linearizing (9) by neglecting $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$, and linearizing (10) by neglecting $\boldsymbol{v}\cdot\nabla h$ and replacing a with a constant value a_0, we have

$$\partial_t \boldsymbol{v} + \nabla\,h = 0, \qquad \partial_t h + a_0^2\,\nabla\cdot\boldsymbol{v} = 0. \tag{17}$$

Eliminating $\boldsymbol{v}$ from the two equations, we obtain the wave equation $(\partial_t^2 - a_0^2\nabla^2)h = 0$ for sound waves. Using it, we obtain the wave equation for $\boldsymbol{v}$ as well. Thus, we have

$$(\partial_t^2 - a_0^2\nabla^2)h = 0, \qquad (\partial_t^2 - a_0^2\nabla^2)\,\boldsymbol{v} = 0. \tag{18}$$

It is remarkable that we have a close analogy between the two systems of fluid and electromagnetism. In vacuum space where $q^{\rm e} = 0$, $\boldsymbol{J}^{\rm e} = 0$, the wave equations (16) reduce to

$$(\partial_t^2 - c^2\,\nabla^2)\,\phi^{(\rm e)} = 0, \qquad (\partial_t^2 - c^2\,\nabla^2)\,\boldsymbol{A} = 0. \tag{19}$$

It is seen that c (light speed) $\Leftrightarrow$ a_0 (sound speed). Notable feature of the sund wave equations (18) is *non-dispersive*. Namely, the dispersion relation for waves of wave number k and frequency ω is given by $\omega^2 = a_0^2 k^2$, and the phase speed ω/k is equal to a_0 independent of the wave length $2\pi/k$. The same is true for the equation (19) of the electromagnetic wave. Note that this is closely related to the system of Maxwell equations.

In addition, the second equation of (17), obtained from the continuity equation by linearization, is analogous to the *Lorentz condition* (15) by the correspondence,

$$(c,\,\boldsymbol{A},\,\phi^{(\rm e)}) \quad \Longleftrightarrow \quad (a_0,\,a_0\boldsymbol{v},\,h).$$

This implies possibility of formulation of *fluid* Maxwell equations for which the vector potential is played by $\boldsymbol{v}$ (except for the coefficient a_0, abbreviated in the present formulation for simplicity) and the scalar potential played by h. According to this finding, let us introduce two fields defined by

$$\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla\,h, \qquad \boldsymbol{H} = \nabla\times\boldsymbol{v}.$$

Consider the following transformations: $\boldsymbol{v}' = \boldsymbol{v} + \nabla f,\ h' = h - \partial_t f$. The vector fields $\boldsymbol{E}'$ and $\boldsymbol{H}'$ defined by $\boldsymbol{v}'$ and h' are unchanged, namely we have $\boldsymbol{E}' = \boldsymbol{E}$ and $\boldsymbol{H}' = \boldsymbol{H}$. Thus in fluid flows, there exists the same gauge invariance as that in the electromagnetism.

3 Fluid Maxwell Equations

Based on the following definition of the two vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$ (analogous to (14)),

$$\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla\,h, \qquad \boldsymbol{H} = \nabla\times\boldsymbol{v}, \tag{20}$$

Fluid Maxwell Equations can be derived from the fluid equations (9) $\sim$ (11) as follows:

$$\text{(A)} \quad \nabla\cdot\boldsymbol{H} = 0, \qquad \text{(B)} \quad \nabla\times\boldsymbol{E} + \partial_t\boldsymbol{H} = 0, \tag{21}$$

$$\text{(C)} \quad \nabla\cdot\boldsymbol{E} = q, \qquad \text{(D)} \quad a_0^2\,\nabla\times\boldsymbol{H} - \partial_t\boldsymbol{E} = \boldsymbol{J}, \tag{22}$$

where
$$q \equiv \nabla\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}], \qquad \boldsymbol{J} \equiv \partial_t^2\boldsymbol{v} + \nabla\partial_t h + a_0^2\nabla\times(\nabla\times\boldsymbol{v}), \tag{23}$$

and a_0 is a constant (the sound speed in undisturbed state). From the calculus ∂_t(C) + div(D) operated on the two equations of (22), we have the *charge conservation*: $\partial_t q + \text{div}\boldsymbol{J} = 0$. Using (9) and the definition $\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h$, the *fluid-electric* field $\boldsymbol{E}$ is given by

$$\boldsymbol{E} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \quad = \boldsymbol{\omega}\times\boldsymbol{v} + \nabla(\tfrac{1}{2}\,v^2). \tag{24}$$

Hence, the *charge density* q is given by

$$q = \nabla\cdot\boldsymbol{E} = \text{div}\big[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\big]. \tag{25}$$

3.1 Derivation

Derivation of the fluid Maxwell equations (21) and (22) is carried out as follows.

(a) Equation (A) is deduced immediately from the definition of $\boldsymbol{H} = \nabla\times\boldsymbol{v} = \boldsymbol{\omega}$.

(b) Equation (B) is an identity obtained from the definition (20). Moreover, if the expression (24) is substituted to $\boldsymbol{E}$ and $\boldsymbol{\omega}$ to $\boldsymbol{H}$, then the equation (B) reduces to the vorticity equation (11).

(c) Equation (C) is just div [Eq.(24)] with q defined by (23).

(d) Equation (D) can be derived in the following way. Applying ∂_t to $\boldsymbol{E} = -\partial_t\boldsymbol{v} - \nabla h$, we obtain

$$-\partial_t\boldsymbol{E} - \partial_t^2\boldsymbol{v} = \nabla\partial_t h,$$

Adding the term $a_0^2\,\nabla\times\boldsymbol{H} = a_0^2\nabla\times(\nabla\times\boldsymbol{v})$ on both sides, this can be rearranged as follows:

$$a_0^2\,\nabla\times\boldsymbol{H} - \partial_t\boldsymbol{E} = \boldsymbol{J}, \qquad \boldsymbol{J} = \partial_t^2\boldsymbol{v} + \nabla\partial_t h + a_0^2\nabla\times(\nabla\times\boldsymbol{v}).$$

which is nothing but the equation (D). The vector $\boldsymbol{J}$ can be given another expression by using $\partial_t h = -(\boldsymbol{v}\cdot\nabla)h - a^2\nabla\cdot\boldsymbol{v}$ from the continuity equation (10):

$$\boldsymbol{J} = \partial_t^2\boldsymbol{v} + a_0^2\nabla\times(\nabla\times\boldsymbol{v}) - \nabla\big(a^2\,\nabla\cdot\boldsymbol{v}\big) - \nabla\big((\boldsymbol{v}\cdot\nabla)h\big).$$

This can be rewritten as $\boldsymbol{J} = (\partial_t^2 - a_0^2\nabla^2)\boldsymbol{v} + \boldsymbol{J}^*$, and $\boldsymbol{J}^* = a_0^2\,\nabla(\nabla\cdot\boldsymbol{v}) - \nabla\big(a^2\,\nabla\cdot\boldsymbol{v}\big) - \nabla(\boldsymbol{v}\cdot\nabla\,h)$, where the following identity is used:

$$\nabla(\nabla\cdot\boldsymbol{v}) = \nabla\times(\nabla\times\boldsymbol{v}) + \nabla^2\boldsymbol{v}, \tag{26}$$

4 Equation of Sound Wave

Suppose that a localized flow is generated at an initial instant in otherwise uniform state at rest, where undisturbed values of the pressure, density and enthalpy are respectively p_0, ρ_0 and h_0. Equation of sound wave is derived from the system of fluid Maxwell equations (A) $\sim$ (D) as follows.

Differentiating Eq.(D) with respect to t, and eliminating $\partial_t\boldsymbol{H}$ by using (B), we obtain

$$\partial_t^2\boldsymbol{E} + a_0^2\nabla\times(\nabla\times\boldsymbol{E}) = -\partial_t\boldsymbol{J}. \tag{27}$$

The second term on the left can be rewritten by using the identity (26), with v replaced by $\boldsymbol{E}$. Then the equation (27) reduces to

$$(\partial_t^2 - a_0^2\nabla^2)(\boldsymbol{E} + \partial_t\boldsymbol{v}) \;=\; -a_0^2\nabla(\nabla\cdot\boldsymbol{E}) - \partial_t\boldsymbol{J}^*, \tag{28}$$

$$\boldsymbol{J}^* = \nabla\big((a_0^2 - a^2)\,\nabla\cdot\boldsymbol{v}\big) - \nabla(\boldsymbol{v}\cdot\nabla\,h) \equiv a_0^2\,\nabla\hat{Q}, \qquad \hat{Q} \,= (1-\hat{a}^2)\nabla\cdot\boldsymbol{v} - a_0^{-2}(\boldsymbol{v}\cdot\nabla)h, \tag{29}$$

where $\hat{a} = a/a_0$. We have $\boldsymbol{E} + \partial_t\boldsymbol{v} = -\nabla\,h$ from (20), and also $\partial_t\boldsymbol{J}^* = a_0^2\,\nabla\partial_t\hat{Q}$ from (29). Therefore, we can integrate (28) spatially, since all the terms are of the form of gradient of scalar fields. Dividing (28) with $-a_0^2$ and integrating it, we obtain the following wave equation:

$$(a_0^{-2}\,\partial_t^{\,2} - \nabla^2)\,\tilde{h} = S(\boldsymbol{x},t), \qquad S(\boldsymbol{x},t) \equiv \nabla\cdot\boldsymbol{E} + \partial_t\hat{Q}, \tag{30}$$

where $\tilde{h} \equiv h - h_0 = (p - p_0)/\rho$. Thus, the vectorial form of wave equation (28) has been reduced to the equation for a scalar field $\tilde{h}$ (see the first of (18)). The term $S(\boldsymbol{x},t)$ is a source of the wave.

Using (24) and (25), we obtain an explicit form of the first $\nabla \cdot \boldsymbol{E}$ of the source S as $\nabla \cdot \boldsymbol{E} = \mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v}) + \nabla^2 \frac{1}{2} v^2$. The first term $\mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v})$ implies that the motion of $\boldsymbol{\omega}$ generates sound waves. This is the source term of the *Vortex sound* (Kambe 2010c), and contribution from the second term $\nabla^2 \frac{1}{2} v^2$ vanishes in an ideal fluid in which total kinetic energy $\int \frac{1}{2} v^2 \, \mathrm{d}^3\boldsymbol{x}$ is conserved. Mach number of the source flow is defined by $M = |\boldsymbol{v}|/a_0$, then the second term $\partial_t \hat{Q}$ of S is $O(M^2)$, namely, higher order if M is small enough.

5 Equation of motion of a test particle in a flow field

Analogy between fluid mechanics and electromagnetism is also found in the equation of motion of a test particle in a flow field as well. Suppose that a *test particle* of mass m is moving in a flow field $\boldsymbol{v}(\boldsymbol{x}, t)$, which is *unsteady, rotational and compressible*. The size of the particle and its velocity are assumed to be so small that its influence on the background velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ is negligible, namely the velocity field $\boldsymbol{v}$ is regarded as independent of the position and velocity of the particle.

The particle velocity is defined by $\boldsymbol{u}(t)$ relative to the fluid velocity $\boldsymbol{v}$. Then, the total particle velocity is $\boldsymbol{u} + \boldsymbol{v}$. In this circumstance, the i-th component of total momentum P_i associated with the test particle moving in the flow field is expressed by the sum: $P_i = m u_i + m_{ik} u_k$,[3] and the equation of motion of the particle is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P} \;=\; m\,\boldsymbol{E} + m\,\boldsymbol{u} \times \boldsymbol{H} - m\,\nabla \phi_g, \tag{31}$$

(Kambe 2010b), where $\boldsymbol{E}$ and $\boldsymbol{H}$ take the same expressions as those of (20), and $\boldsymbol{P} = (P_i)$, $P_i = m u_i + m_{ik} u_k$, and $\phi_g = g z$. Obviously, the equation (31) is analogous to the equation of motion of a *charged* particle in an electromagnetism of electric field $\boldsymbol{E}^{\mathrm{em}}$ and magnetic field $\boldsymbol{H}^{\mathrm{em}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}_e) = e\,\boldsymbol{E}^{\mathrm{em}} + (e/c)\,\boldsymbol{v}_e \times \boldsymbol{H}^{\mathrm{em}} - m\nabla\Phi_g\,, \tag{32}$$

where $\boldsymbol{v}_e$ is the velocity of a charged particle, and c the light velocity. Rewriting the second term of (31) as $(m/a_0)\,\boldsymbol{u} \times (a_0\boldsymbol{H})$, and comparing the first two terms on the right of (31) and (32), it is found that there is correspondence: $e \Leftrightarrow m$, $\boldsymbol{E}^{\mathrm{em}} \Leftrightarrow \boldsymbol{E}$, and $\boldsymbol{H}^{\mathrm{em}} \Leftrightarrow a_0\,\boldsymbol{H}$.

6 Summary

It is shown that there exist similarities between electromagnetism and fluid mechanics. The correspondence betwenn them permits formulation of a system of fluid Maxwell equations. It is found that the sound wave in the fluid is analogous to the electromagnetic wave, in the sense that phase speeds of both waves are independent of wave lengths, *i.e.* non-dispersive. Another analogy between fluid mechanics and electromagnetism is presented for the equations of motion of a test particle in flow field and in electromagnetic field.

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[3]According to the hydrodynamic theory (*e.g.* Landau & Lifshitz (1987)) when a solid particle moves through the fluid (at rest), the fluid energy induced by the relative particle motion of velocity $\boldsymbol{u} = (u_i)$ is expressed in the form $\frac{1}{2}\,m_{ik} u_i u_k$ by using the *mass tensor* m_{ik}. Additional fluid momentum induced by the particle motion is given by $m_{ik} u_k$.