COVARIANT PERTURBATIONS THEORY IN GENERAL MULTI-FLUIDS COSMOLOGY

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Abstract
We develop a variational approach, inspired by the work of Maldacena (Maldacena 2003), to study covariant cosmological perturbations in general multi-fluids extended gravity. A special attention is paid to the minimization of the propagating degrees of freedom to obtain simple equations of motion. In particular, by parametrizing perfect fluids with scalar fluids and by working within the ADM formalism, we manage to introduce new gauge invariant quantities, generalizing Mukhanov-Sasaki variables. Those developments are crucial to deeply understand the influence of general extended gravity either in the inflationary epoch or in the late-time inflation. In this proceeding, we present only the quadratic Lagrangian at first order for scalar perturbations deduced from the gravitational constraint equations.

1. Introduction
The predictions of General Relativity are in good agreement with the cosmological and astronomical observations only if we suppose the evolution of our Universe is driven by unknown forms of energy: Dark matter and Dark energy. Dark matter dominates the gravitational attraction at small scales and seems well described by non-relativistic particles, weakly coupled to baryons. At larger scales, the evolution of the Universe can be associated with a form of energy behaving like a cosmological constant.
The interpretation of dark energy purely in term of a new constant of Nature suffers however several problems. To go beyond this simple proposal, we can make this cosmological constant vary, thus introducing a scalar field.
The main goal of this proceeding is to describe **in full generality** the equations of motion for wide classes of dark energy models i.e. dark energy as a scalar field (non-)minimally coupled to multiple matter components.
In the observable frame, named Dicke-Jordan frame, the action we consider writes down:

$$S_{BD} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\phi}{2\kappa^2}\tilde{R} - \frac{\omega(\phi)}{2\kappa^2\phi}\partial_\mu\phi\partial_\upsilon\phi + \tilde{U}(\phi) + \sum_i \tilde{P}(\tilde{X}_{\psi_i})\right] \tag{1}$$

with $\kappa^2 = 8\pi G$ the gravitational coupling and $\omega(\phi)$ the Dicke-Jordan coupling function.

2. Perfect fluid as a scalar field
To minimize the degrees of freedom, we have to parametrize the matter part of action (1).
A barotropic and irrotational perfect fluid propagates only one degree of freedom. Therefore, at Lagrangian level, the question of the treatment of perfect fluids in terms of scalar fields can be asked. The answer is positive as shown in (Boubekeur 2008). In the following, we develop this formulation in the Dicke-Jordan frame (i.e. observational frame).
The energy-momentum tensor of a perfect fluid reads:

$$\tilde{T}_{\mu\upsilon} = (\tilde{\rho} + \tilde{p})\tilde{u}_\mu\tilde{u}_\upsilon + \tilde{p}\tilde{g}_{\mu\upsilon} \tag{2}$$

where $\tilde{\rho}$ and $\tilde{p}$ are respectively the energy density and pressure of the fluid in the Dicke-Jordan, while $\tilde{u}_\mu$ is the fluid four-velocity.

Introducing the Lagrangian $L = \tilde{P}(\tilde{X})$ with $\tilde{X} \equiv -\tilde{g}^{\mu\upsilon}\partial_\mu\psi\partial_\upsilon\psi$, associated to the pressure of a scalar field, and varying it with respect to the metric, we find the stress-energy tensor:

$$\tilde{T}_{\mu\upsilon} = 2\tilde{P}(\tilde{X})_{,X}\partial_\mu\psi\partial_\upsilon\psi + \tilde{P}(\tilde{X})\tilde{g}_{\mu\upsilon} \tag{3}$$

This tensor looks quite similar to eq.2. An identification can be done in the following way:

$$\tilde{\rho} = 2\tilde{X}\tilde{P}_{,X} - \tilde{P}\ ,\ \tilde{p} = \tilde{P} \text{ and } \tilde{u}_\mu = \frac{\partial_\mu \psi}{\sqrt{\tilde{X}}}\ . \tag{4}$$

Specializing to a barotropic fluid i.e. $\tilde{p} = w\tilde{\rho}$, we can explicitly write the pressure and the energy density in terms of $\tilde{X}$ alone (At homogeneous order: $p_i = A_i^{2(2-\alpha)}\dot{\psi}_i^{2\alpha}$) :

$$\tilde{P} = \tilde{X}^\alpha \text{ with } \alpha = \frac{1+\omega}{2\omega} \tag{5}$$

for $w \neq 0$. However, as proven in (Boubekeur 2008), the limit $w \to 0$ can be taken at the level of the equations of motion and corresponds to the case of dust (baryons or cold dark matter). This case is of special interest since it describes the energy density component of baryons and cold dark matter in the late time Universe.

In this proceeding, we are interested in the case in which a scalar field ϕ is, in full generality, non-minimally coupled to gravity. In this case, the action in Einstein-Hilbert frame can be rewritten via a conformal transformation $\tilde{g}_{\alpha\beta} = A_\psi^2(\varphi) g_{\alpha\beta}$ with $A_i(\varphi)$ the conformal coupling:

$$S_{EH} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + \frac{1}{2}(\partial\varphi)^2 - V(\varphi) + \sum_i P(X_i, A_i)\right] \tag{6}$$

with $\kappa^2 = 8\pi G$ the gravitational coupling and $\omega(\phi)$ the Dicke-Jordan coupling function.

3. Constraints at homogeneous order

The consistency of our approach can be checked through the derivation of the Friedmann and conservation equations. Since the Friedmann equations are resulting of the variation of the action with respect to the metric $g_{\mu\nu}$, they remained unchanged under our ersatz.

We assume a flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with metric:

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j \tag{7}$$

Using this metric, action (6) and the perfect fluid ersatz, we find the Klein-Gordon equation for an interacting scalar fluid and a set of equations describing the motion of various 'matter' scalar fields.

$$\begin{aligned} &\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} + \sum_i \frac{d\ln A_i^{2(2-\alpha)}}{d\varphi}(\rho_i - p_i) = 0 \\ &(2\alpha-1)\ddot{\psi}_i + 3H\dot{\psi}_i + \frac{d\ln A_i^{2(2-\alpha)}}{dt}\dot{\psi}_i = 0 \end{aligned} \tag{8}$$

For simplicity, in this proceeding[1], we suppose all the 'matter' fields have the same physical origin: we consider only one α parameter that has the same equation of state for all matter components. However, every component has its own coupling function A_i to gravity.

4. Toward observables at linear order

4.1. Linear perturbations

To express observables at linear order, we split our perturbed metric using the ADM formalism:

$$ds^2 = -N(t)^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{9}$$

Thanks to diffeomorphism invariance, we can choose a gauge such that: $N = 1 + \delta N$, $N^i = a^{-2}\partial_i\beta$, $h_{ij} = a^2e^{2\xi}\delta_{ij}$ and $\delta\varphi = 0$. As soon as we're dealing with perturbations in cosmology, the question of

1 The full case is described in (Alimi 2012)

the gauge is crucial. In fact, two gauge choices are preferred in our formalism: the unperturbed scalar field gauge ($\delta\varphi = 0$) and the spatially flat gauge ($\zeta = 0$). However, since the scalar field φ is non-minimally coupled to all other 'matter' scalar fields, it plays a very specific role. Therefore, the gauge comoving with the scalar field is the most natural gauge choice. In this gauge, the action writes down:

$$S = \int d^4x \sqrt{\det(h_{ij})} \left\{ \frac{1}{2\kappa^2} \left[N^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2) \right] + N^{-1}\frac{\dot{\varphi}}{2} - NV(\varphi) + N\sum_i P(X_i, A_i) \right\} \qquad (10)$$

The main feature of the ADM formalism is the non-propagation of the metric components N and N^i. Instead of having two propagating equations, we obtain the Hamiltonian and momentum constraints as follows:

$$\begin{aligned} \delta N &= \frac{\dot{\zeta}}{H} + \frac{\kappa^2}{H} \sum_i \alpha P_i \frac{\delta\psi}{\dot{\psi}} \\ H\partial^2\beta + \partial^2\zeta - 3H\dot{\zeta} &= \sum_i \alpha(1-2\alpha)\kappa^2 P_i \frac{\delta\dot{\psi}}{\dot{\psi}} + \sum_i (1-\alpha)(1-2\alpha)\kappa^2 P_i \delta N - \kappa^2 \delta NV \end{aligned} \qquad (11)$$

Substituting these two constraints into the action (10), we find the quadratic action on linear perturbations.
Please note that a dot is a derivative with respect to time whereas, in the following, a prime denotes a derivative with respect to conformal time.

4.2. Quadratic action at first order

During the computation of the quadratic action at first order, a generalized form of the Mukhanov-Sasaki variables, widely used in inflation, arises. Those quantities, formally equivalent to the potential of the velocity fields, are gauge invariants:

$$G_0 = Hz_0\left(\frac{\zeta}{H} - \frac{\delta\varphi}{\dot{\varphi}}\right) \text{ and } G_i = Hz_i\left(\frac{\zeta}{H} - \frac{\delta\psi_i}{\dot{\psi}_i}\right) \text{ with } z_\mu = \kappa a \frac{\sqrt{\rho_\mu + p_\mu}}{H} \qquad (12)$$

Therefore, the gauge invariant quadratic action at first order is given in Figure 1. The quantities $\tilde{H}_i$, A_i , C_i and D_i determining the evolution of the perturbations are only functions of the background variables and the couplings.

$$\begin{aligned} S = &\frac{1}{2}\int d^4x G_0'^2 - (\partial G_0)^2 + G_0^2\left[\frac{z_0''}{z_0} + \sum_i (2\alpha-1)\frac{z_i^2}{z_0^2}\left(A_i + \frac{\alpha\kappa^2 V}{2\alpha-1}D_i\right)\right] \\ &+\frac{1}{2}\int d^4x\,(2\alpha-1)\sum_i\left[G_i'^2 - c_s^2(\partial G_i)^2 - G_i^2\left(\frac{\alpha-2}{2\alpha-1}\tilde{\mathcal{H}}_i' - \left(\frac{\alpha-2}{2\alpha-1}\right)^2\tilde{\mathcal{H}}_i^2\right)\right] \\ &+\frac{1}{2}\int d^4x\,(2\alpha-1)\sum_i 4G_0'G_i\frac{z_i}{z_0}\left(\frac{z_i'}{z_i} - \mathcal{H} + \frac{\alpha\kappa^2 V}{2\alpha-1}\frac{a^2}{\mathcal{H}}\right) \\ &+\frac{1}{2}\int d^4x\,(2\alpha-1)\sum_i 2G_0G_i\frac{z_i}{z_0}\left[C_i + \frac{\alpha\kappa^2 V}{2\alpha-1}D_i\right] \\ &-\frac{1}{2}\int d^4x\,\kappa^4\alpha V\sum_{i,j}G_iG_jz_iz_j \end{aligned}$$

Usual mass of the scalar field in non-minimally coupled models.

Mass of the *'matter'* scalar fields (non-fixed α) in non-minimal models.

Couplings of the *'matter'* scalar fields and the scalar field φ : due to the interactions with the potential and the non-minimal couplings.

Couplings linked with our formalism and reabsorbed in the equations of motion.

Figure 1: Quadratic action at first order.

References

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